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The Projective Geometry of Ambiguous Surfaces

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The projective geometry of ambiguous surfaces

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The projective geometry underlying the ambiguous case of scene reconstruction from image correspondences is developed. The ambiguous case arises when reconstruction yields two or more essentially different surfaces in space, each capable of giving rise to the image correspondences. Such surfaces naturally occur in complementary pairs. Ambiguous surfaces are examples of rectangular hyperboloids. Complementary ambiguous surfaces intersect in a space curve of degree four, which splits into two components, namely a twisted cubic (space curve of degree three), and a straight line.

For each ambiguous surface compatible with a given set of image correspondences, a complementary surface compatible with the same image correspondences can always be found such that both the original surface and the twisted cubic contained in the intersection of the two surfaces are invariant under the same rotation through 180°. In consequence, each ambiguous surface is subject to a cubic polynomial constraint. This constraint is the basis of a new proof of the known result that there are, in general, exactly ten scene reconstructions compatible with five given image correspondences.

Ambiguity also arises in reconstruction based on image velocities rather than on image correspondences. The two types of ambiguity have many similarities because image velocities are obtained from image correspondences as a limit, when the distances between corresponding points become small. It is shown that the amount of similarity is restricted, in that when passing from image correspondences to image velocities, some of the detailed geometry of the ambiguous case is lost.

1. Introduction

The possibility of obtaining information about the shape of the environment from the correspondences between two images first arose during the 19th century with the invention of photography. The methods developed for exploiting this possibility were based on projective geometry since projection provides a good model for image formation. At first the shape of the environment was reconstructed from image correspondences by linear methods (Sturm 1869), which did not make full use of the rigidity of the environment. Later methods incorporated the rigidity constraint (Faugeras & Maybank 1989; Kruppa 1913), thus allowing reconstruction with fewer image correspondences, but at the cost of greatly increasing the complexity of the reconstruction algorithm.

More recently methods for reconstruction have been transformed by the advent of electronic cameras and computers. Large numbers of images are obtained by an electronic camera in a short space of time, the image correspondences are found automatically, and reconstruction is carried out by a computer algorithm. In this way a robot or an automatic vehicle can obtain useful information by passive means well suited to a wide range of environments. The modern approach to reconstruction is based on euclidean geometry and the vector calculus, rather than on projective geometry (Longuet-Higgins 1981; Tsai & Huang 1984). A very large number of algorithms for reconstruction have been published in the computer vision literature. For example, linear algorithms have been constructed by Longuet-Higgins (1981), and Tsai & Huang (1984). Examples of nonlinear algorithms are given by T. Buchanan (unpublished work, 1987) and Horn (1990).

Experiments by Fang & Huang (1984) have shown that reconstruction is often unstable, in that small changes in the positions of corresponding image points lead to large changes in the reconstructed scene points. An extreme case of instability arises when a set of image correspondences is ambiguous, in that the set is compatible with two or more essentially different reconstructions. The ambiguous case can thus be regarded as a first approximation to the more common unstable case. The ambiguous case has the advantage of being accessible to mathematical analysis. At the same time it contains the essential features of the more general unstable case.

Ambiguity has been studied previously using both the older projective goemetric approach (Faugeras & Maybank 1989; Hofmann 1950; Kruppa 1913; Sturm 1869), and the newer euclidean approach (Horn 1990; Longuet-Higgins 1988; Negahdaripour 1989). In this paper the projective geometric approach to ambiguity is developed in detail, using the work of Buchanan (1987), Hofmann (1950) and Wunderlich (1942) as a starting point. A number of new results are obtained, the most important among these being a cubic polynomial constraint on ambiguous surfaces. This constraint is the basis of a new proof of Demazure's (1988) result that there are in general exactly ten camera displacements compatible with five given image correspondences. It is thought that the results on ambiguity described in this paper will lead to a better understanding of the instabilities arising in reconstruction based on image correspondences.

1.1. *Overview*

In §2 two formulations of the reconstruction problem are described, based respectively on euclidean geometry and projective geometry. Ambiguous surfaces are defined, and some of the basic properties of ambiguous surfaces are obtained. The projective geometry required in later sections is introduced in §3 and §4. In particular, involutions are discussed in detail, since they play a key role in the study of ambiguity. The properties of rectangular quadrics are discussed in §5, with particular emphasis on the rigid involutions of rectangular quadrics.

Certain space curves of degree three naturally arise in the study of ambiguity. These are the horopter curves described in §6. It is shown that each horopter curve is invariant under a unique non-trivial rigid involution. The connections between rigid involutions of horopter curves and rigid involutions of ambiguous surfaces are described.

The results of §5 and §6 are applied in §7. It is shown that an ambiguous surface is invariant under a rigid involution that interchanges the two possible camera positions for the second image. In consequence, ambiguous surfaces are subject to a cubic polynomial constraint. In §8 this constraint is used to show that there are, in general, exactly ten camera displacements compatible with five given image correspondences. The analogous results for reconstruction based on image velocities rather than image displacements are discussed in §9. It is shown that the cubic constraint on ambiguous surfaces still applies, but in a much simpler form. Much of the detailed geometry associated with ambiguity is lost on passing image velocities. Some concluding remarks are made in §10.

The results obtained in §§2–5 are not new, although to the author's knowledge they have not previously been gathered into a single convenient reference. The results of §6 are also known, with the possible exception of the classification of horopter curves contained in an ambiguous surface and invariant under the involution τ_ψ. Most of the results obtained in §§7–9 are new. Among these, the most important is the cubic constraint on ambiguous surfaces obtained in theorems 7.3 to 7.5.

1.2. *Notation*

The notation follows that of Semple & Kneebone (1952). Euclidean three-dimensional space $\mathbb{R}^3$ is regarded as a subset of projective space $\mathbb{P}^3$. Points of $\mathbb{R}^3$ have coordinates $[x_1, x_2, x_3]$, and points of $\mathbb{P}^3$ have coordinates (x_1, x_2, x_3, x_4). The embedding $\mathbb{R}^3 \subset \mathbb{P}^3$ is the usual one

$$[x_1, x_2, x_3] \mapsto (x_1, x_2, x_3, 1).$$

The set $\mathbb{P}^3 \backslash \mathbb{R}^3$ is a plane, Π_∞, known as the plane at infinity. The coordinates x_i of $\mathbb{R}^3$ and $\mathbb{P}^3$ are usually chosen to be rectangular, in which case they are referred to as cartesian coordinates.

Points of $\mathbb{P}^3$ are typically denoted by $\boldsymbol{o}, \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{x}, \boldsymbol{y}$, and lines of $\mathbb{P}^3$ are denoted by g, h, k. The line joining two distinct points $\boldsymbol{a}, \boldsymbol{b}$ of $\mathbb{P}^3$ is $\langle \boldsymbol{a}, \boldsymbol{b} \rangle$, and similarly, the plane containing the line g and the point $\boldsymbol{a}$ not on g is $\langle g, \boldsymbol{a} \rangle$. This notation is extended to an arbitrary number of lines and points. For example, $\langle g, h, \boldsymbol{a} \rangle$ is the smallest subspace of $\mathbb{P}^3$ containing the lines g, h and the point $\boldsymbol{a}$. Occasionally lines are given in the parametrized form

$$t \mapsto \boldsymbol{a} + t\boldsymbol{b},$$

where t is a scalar, and $\boldsymbol{a}, \boldsymbol{b}$ are fixed points. Planes in $\mathbb{P}^3$ are typically denoted by Π, Φ, Ξ; quadric surfaces are denoted by ψ, ϕ; conics are denoted by s; and twisted cubics (space curves of degree three) are denoted by c.

Let a point $\boldsymbol{o}$ of $\mathbb{R}^3$ be fixed as an origin. Then each point $\boldsymbol{x}$ of $\mathbb{R}^3$ defines a vector, namely, the line segment from $\boldsymbol{o}$ to $\boldsymbol{x}$. This vector is denoted by the same symbol $\boldsymbol{x}$ as the point $\boldsymbol{x}$. The dot product $\boldsymbol{x} \cdot \boldsymbol{y}$ and the vector product $\boldsymbol{x} \times \boldsymbol{y}$ of vectors $\boldsymbol{x}, \boldsymbol{y}$ are formed in the usual way (Sokolnikoff & Redheffer 1966). Each non-zero vector $\boldsymbol{x} = (x_1, x_2, x_3, x_4)$ corresponds to a unique point $(x_1, x_2, x_3, 0)$ of Π_∞ known as the direction of $\boldsymbol{x}$. The points of Π_∞ can thus also be regarded as vectors. If $\boldsymbol{x}, \boldsymbol{y}$ are points of Π_∞ then $\boldsymbol{x} \times \boldsymbol{y}$ is defined to be the point of Π_∞ corresponding to the direction orthogonal to the directions of $\boldsymbol{x}$ and $\boldsymbol{y}$.

The tensor product of two vectors $\boldsymbol{x}, \boldsymbol{y}$ is denoted by $\boldsymbol{x} \otimes \boldsymbol{y}$. In applications of the tensor product $\boldsymbol{x}$ and $\boldsymbol{y}$ are points of $\mathbb{P}^3$ contained in the plane $x_4 = 0$. The product $\boldsymbol{x} \otimes \boldsymbol{y}$ is defined to be the 3×3 matrix with i,jth entry equal to $x_i y_j$ for $1 \leqslant i, j \leqslant 3$. As the coordinates of $\boldsymbol{x}$ and $\boldsymbol{y}$ are only defined up to a non-zero scalar multiple, $\boldsymbol{x} \otimes \boldsymbol{y}$ is only defined up to a non-zero scalar multiple.

Invertible linear transformations (collineations) of projective space are typically denoted by ω, and involutions are typically denoted by σ, τ. The value of ω at a point $\boldsymbol{x}$ is $\omega\boldsymbol{x}$ (without brackets). If S is a set of points, for example a line, or a plane, then $\omega(S)$ (with brackets) is the set of $\omega\boldsymbol{x}$ as $\boldsymbol{x}$ ranges over S. If $\omega(S) \subset S$ then S is said to be invariant under ω.

2. Scene reconstruction

In the reconstruction problem, two images of the same set of scene points $\boldsymbol{p}_i$ are taken from distinct projection points, $\boldsymbol{o}$ and $\boldsymbol{a}$, as illustrated in figure 1. The $\boldsymbol{p}_i$ are assumed to be fixed rigidly with respect to $\boldsymbol{o}$ and $\boldsymbol{a}$. The point $\boldsymbol{o}$ is referred to as the optical centre of the first camera, and $\boldsymbol{a}$ is referred to as the optical centre of the second camera. The imaging surface of each camera is the unit sphere with centre at

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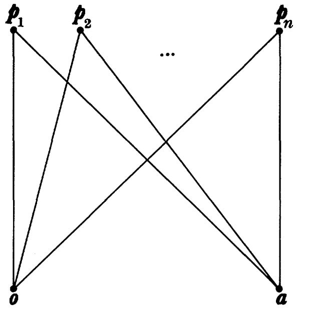

Figure 1. Two views of a set of points in space.

the optical centre of the camera, and the image is formed on the imaging surface by polar projection. Each scene point $\boldsymbol{p}_i$ gives rise to corresponding points $\boldsymbol{q}_i, \boldsymbol{q}'_i$ in the first and second images respectively. The correspondence between $\boldsymbol{q}_i$ and $\boldsymbol{q}'_i$ is denoted by $\boldsymbol{q}_i \leftrightarrow \boldsymbol{q}'_i$.

In practice, the camera projection is more complicated than polar projection onto the unit sphere. This discrepancy between theory and experiment is overcome by calibrating the camera. The acquired image is then transformed in order to obtain the image that would have arisen from polar projection.

2.1. *Euclidean treatment of ambiguity*

Each camera has associated to it a coordinate frame in which the positions of the image points are measured. The displacement of the second camera with respect to the first is specified by giving the translation vector from the first optical centre $\boldsymbol{o}$ to the second optical centre $\boldsymbol{a}$, and the rotation R needed to bring the two camera coordinate frames into alignment after carrying out the translation. It is assumed that $\det(R) = 1$, to exclude the possibility that the two coordinate frames differ by a reflection. When specifying the relative displacement of the cameras by a pair $\{R, \boldsymbol{a}\}$ it is assumed that cartesian coordinates have been chosen with the origin $\boldsymbol{o}$ at the optical centre of the first camera. The translation vector $\boldsymbol{a}$ is the identified with the point $\boldsymbol{a}$ at the optical centre of the second camera. If $\{R, \boldsymbol{a}\}$ is known then the positions $\boldsymbol{p}_i$ of the scene points relative to the two cameras are easily calculated from the image correspondences. The reconstruction problem thus reduces to the problem of recovering $\{R, \boldsymbol{a}\}$ from the image correspondences.

Let the coordinates of the image points $\boldsymbol{q}_i$ and $\boldsymbol{q}'_i$ be measured in the appropriate camera coordinate frame, and let p_i, p'_i be scalars such that the position of $\boldsymbol{p}_i$ relative to $\boldsymbol{o}$ is $p_i \boldsymbol{q}_i$, and such that the position of $\boldsymbol{p}_i$ relative to $\boldsymbol{a}$ is $p'_i \boldsymbol{q}'_i$. It follows from the definition of $\{R, \boldsymbol{a}\}$ that

$$p'_i \boldsymbol{q}'_i = R(p_i \boldsymbol{q}_i - \boldsymbol{a}). \tag{1}$$

A precise formulation of the reconstruction problem is now possible: given n image correspondences $\boldsymbol{q}_i \leftrightarrow \boldsymbol{q}'_i$, find $\{R, \boldsymbol{a}\}$ and p_i, p'_i such that (1) holds for each i, $1 \leqslant i \leqslant n$.

The well-known scaling ambiguity inherent in reconstruction (Horn 1990) is apparent from (1). If p_i, p'_i and $\boldsymbol{a}$ are scaled by the same non-zero constant λ, then

$$\lambda p'_i \boldsymbol{q}'_i = R(\lambda p_i \boldsymbol{q}_i - \lambda \boldsymbol{a}),$$

thus $\{R, \boldsymbol{a}\}$ and $\{R, \lambda \boldsymbol{a}\}$ are both compatible with the same image correspondences $\boldsymbol{q}_i \leftrightarrow \boldsymbol{q}'_i$. The camera displacements $\{R, \boldsymbol{a}\}$ and $\{R, \lambda \boldsymbol{a}\}$ cannot be distinguished using only image correspondences. The scaling ambiguity is a mathematical formulation of the fact that a large object far from the camera gives rise to the same image as a similar but smaller object near to the camera. In view of the ever present scaling ambiguity, the camera displacements $\{R, \boldsymbol{a}\}$ and $\{R, \lambda\boldsymbol{a}\}$ are counted together as a single solution to the reconstruction problem. If it is necessary to specify a unique value of $\boldsymbol{a}$, then arbitrary conditions such as $\boldsymbol{a}\cdot\boldsymbol{a} = 1$, $a_3 > 0$ can be imposed.

Ambiguity in reconstruction is distinct from the ubiquitous scaling ambiguity. Ambiguity in reconstruction arises when two camera displacements $\{R, \boldsymbol{a}\}$, $\{S, \boldsymbol{b}\}$ are compatible with the same image correspondences, even though $\lambda\boldsymbol{a} \neq \boldsymbol{b}$ for all λ. In discussing ambiguity it is convenient to make the following definition.

Definition 1.1. Two camera displacements $\{R, \boldsymbol{a}\}$ and $\{S, \boldsymbol{b}\}$ are essentially different if $\boldsymbol{a}\times\boldsymbol{b} \neq 0$.

Longuet-Higgins (1988) shows that if $\{R, \boldsymbol{a}\}$ and $\{S, \boldsymbol{b}\}$ are camera displacements compatible with the same image correspondences such that $\boldsymbol{a}\times\boldsymbol{b} = 0$, then either $R = S$, or $R = \sigma S$ where σ is a rotation of 180° about the axis $R\boldsymbol{a}$. The displacements $\{R, \boldsymbol{a}\}$, $\{\sigma R, \boldsymbol{a}\}$ are called a twisted pair (Horn 1990). In this paper twisted pairs are counted as a single solution to the reconstruction problem. Thus when considering two solutions $\{R, \boldsymbol{a}\}$, $\{S, \boldsymbol{b}\}$ to the reconstruction problem it is always assumed that $\boldsymbol{a}\times\boldsymbol{b} \neq 0$.

The equation of an ambiguous surface is obtained using the method given by Longuet-Higgins (1988). Let $\{R, \boldsymbol{a}\}$, $\{S, \boldsymbol{b}\}$ be two essentially different camera displacements compatible with the same set of image correspondences $\boldsymbol{q}\leftrightarrow\boldsymbol{q}'$. (The subscript i is omitted from now on.) It follows from (1) that there exist scalars p_1, p_1' and p_2, p_2' such that

$$p_1'\boldsymbol{q}' = R(p_1\boldsymbol{q}-\boldsymbol{a}), \tag{2}$$

$$p_2'\boldsymbol{q}' = S(p_2\boldsymbol{q}-\boldsymbol{b}). \tag{3}$$

As $\boldsymbol{q}$ varies the point $p_1\boldsymbol{q}$ traces out a surface compatible with a displacement of the optical centre of the camera from $\boldsymbol{o}$ to $\boldsymbol{a}$; and similarly, as $\boldsymbol{q}$ varies the point $p_2\boldsymbol{q}$ traces out a surface compatible with a displacement of the optical centre of the camera from $\boldsymbol{o}$ to $\boldsymbol{b}$. The two surfaces formed in this way are known as complementary ambiguous surfaces.

The vector product of (2) and (3) yields

$$R(p_1\boldsymbol{q}-\boldsymbol{a})\times S(p_2\boldsymbol{q}-\boldsymbol{b}) = 0. \tag{4}$$

The scalar product of (4) with $S\boldsymbol{b}$ is homogeneous in p_2. After cancelling p_2 from the scalar product the following expression for p_1 is obtained:

$$p_1 = (R\boldsymbol{a}\times S\boldsymbol{q})\cdot S\boldsymbol{b}/(R\boldsymbol{q}\times S\boldsymbol{q})\cdot S\boldsymbol{b}. \tag{5}$$

Similarly, the scalar product of (4) with $R\boldsymbol{a}$ yields

$$p_2 = (S\boldsymbol{b}\times R\boldsymbol{q})\cdot R\boldsymbol{a}/(S\boldsymbol{q}\times R\boldsymbol{q})\cdot R\boldsymbol{a}. \tag{6}$$

On setting $\boldsymbol{x} = p_1\boldsymbol{q}$ in (5) and $\boldsymbol{x} = p_2\boldsymbol{q}$ in (6) the following equations for the complementary ambiguous surfaces are obtained.

$$(R\boldsymbol{x}\times S\boldsymbol{x})\cdot S\boldsymbol{b} = (R\boldsymbol{a}\times S\boldsymbol{x})\cdot S\boldsymbol{b}, \quad (S\boldsymbol{x}\times R\boldsymbol{x})\cdot R\boldsymbol{a} = (S\boldsymbol{b}\times R\boldsymbol{x})\cdot R\boldsymbol{a}.$$

On setting $U = S^{\mathrm{T}}R$, the equations for the complementary ambiguous surfaces become

$$(U\boldsymbol{x}\times\boldsymbol{x})\cdot\boldsymbol{b} = (U\boldsymbol{a}\times\boldsymbol{x})\cdot\boldsymbol{b}, \tag{7}$$

$$(U^{\mathrm{T}}\boldsymbol{x}\times\boldsymbol{x})\cdot\boldsymbol{a} = (U^{\mathrm{T}}\boldsymbol{b}\times\boldsymbol{x})\cdot\boldsymbol{a}. \tag{8}$$

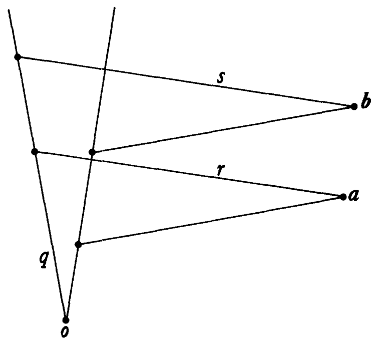

Figure 2

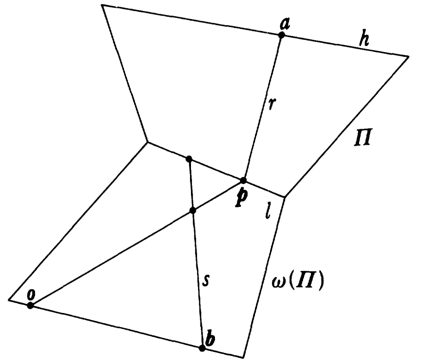

Figure 3

Figure 2. The geometry of ambiguity.

Figure 3. Geometric construction of an ambiguous surface.

The surfaces (7) and (8) are denoted respectively by ψ and ϕ. Each surface is of degree two. The surfaces ψ and ϕ both contain the origin $\boldsymbol{o}$, and the two possible optical centres $\boldsymbol{a}$ and $\boldsymbol{b}$ of the second camera.

Equations (7) and (8) are in a form appropriate for points $\boldsymbol{x} = [x_1, x_2, x_3]$ in $\mathbb{R}^3$. The corresponding equations for projective space $\boldsymbol{P}^3$ are obtained by introducing a new coordinate x_4, and writing (7) and (8) in homogeneous form (Walker 1962). For example, (7) becomes

$$(U\boldsymbol{x} \times \boldsymbol{x})\cdot\boldsymbol{b} = x_4(U\boldsymbol{a} \times \boldsymbol{x})\cdot\boldsymbol{b}. \tag{9}$$

Equation (7) is recovered from (9) on setting $x_4 = 1$. It follows from (9) that the conic formed by the intersection of ψ with the plane at infinity, $x_4 = 0$, is given by

$$(U\boldsymbol{x} \times \boldsymbol{x})\cdot\boldsymbol{b} = 0. \tag{10}$$

2.2. *Projective geometric treatment of ambiguity*

The projective geometric formulation of the reconstruction problem is as follows. As in §2.1, let two images of the same scene be obtained from cameras with optical centres at distinct points $\boldsymbol{o}$ and $\boldsymbol{a}$. A point $\boldsymbol{q}$ in the first image defines a projection line $\langle \boldsymbol{o}, \boldsymbol{q}\rangle$ such that all points on this line project to $\boldsymbol{q}$, and similarly, a point $\boldsymbol{q}'$ in the second image defines a projection line $\langle \boldsymbol{a}, \boldsymbol{q}'\rangle$ such that all points on this line project to $\boldsymbol{q}'$. Thus image formation is modelled as a linear transformation from the points of $\mathbb{P}^3$ to the two-dimensional projective space of lines (i.e. sight rays) passing through the optical centre of the camera. The lines $\langle \boldsymbol{o}, \boldsymbol{q}\rangle$ and $\langle \boldsymbol{a}, \boldsymbol{q}'\rangle$ correspond if and only if they intersect at a scene point $\boldsymbol{p}$.

In the ambiguous case there exist points $\boldsymbol{a}, \boldsymbol{b}$ on distinct lines through $\boldsymbol{o}$, such that the camera taking the second image can have its optical centre either at $\boldsymbol{a}$ or at $\boldsymbol{b}$. Let q be the line of points projecting to a point $\boldsymbol{q}$ in the first image. Let r be the line of points projecting to the corresponding point $\boldsymbol{q}'$ in the second image when the optical centre of the second camera is at $\boldsymbol{a}$, and let s be the line of points projecting to $\boldsymbol{q}'$ in the second image when the optical centre of the second camera is at $\boldsymbol{b}$. Then q is the unique common transversal of r and s passing through $\boldsymbol{o}$, as illustrated in figure 2.

The image taken by the second camera is unchanged whether the camera is thought to be at $\boldsymbol{a}$ or $\boldsymbol{b}$, thus the angles between pairs of lines r_i, r_j through $\boldsymbol{a}$ are equal to the angles between pairs of lines s_i, s_j through $\boldsymbol{b}$ associated with the same image points as r_i, r_j. It follows that there is a linear orthogonal (i.e. angle preserving) transformation ω from the lines through $\boldsymbol{a}$ to the lines through $\boldsymbol{b}$ such that $\omega r = s$.

The linear transformation ω is the basis of a geometrical method for constructing ambiguous surfaces given by Wunderlich (1942), and illustrated in figure 3. The transformation ω induces a linear transformation, also denoted by ω, from the planes through $\boldsymbol{a}$ to the planes through $\boldsymbol{b}$, such that if a line r through $\boldsymbol{a}$ is contained in a plane Π then ωr is contained in $\omega(\Pi)$. Let h be the line through $\boldsymbol{a}$ such that $\omega h = \langle \boldsymbol{o}, \boldsymbol{b} \rangle$. If Π is any plane containing h, then $\omega(\Pi)$ contains $\langle \boldsymbol{o}, \boldsymbol{b} \rangle$. As Π varies through the one-dimensional projective space of planes containing h, $\omega(\Pi)$ varies through the one-dimensional projective space of planes containing $\langle \boldsymbol{o}, \boldsymbol{b} \rangle$. The line $l = \Pi \cap \omega(\Pi)$ sweeps out the ambiguous surface, ψ, associated with the camera displacement from $\boldsymbol{o}$ to $\boldsymbol{a}$.

To prove that l sweeps out ψ, let $\boldsymbol{p}$ be any point of l, let $r = \langle \boldsymbol{a}, \boldsymbol{p} \rangle$, and let s be the line ωr through $\boldsymbol{b}$. Then s and $\langle \boldsymbol{o}, \boldsymbol{p} \rangle$ are contained in $\omega(\Pi)$, thus s intersects $\langle \boldsymbol{o}, \boldsymbol{p} \rangle$. It follows that $\langle \boldsymbol{o}, \boldsymbol{p} \rangle$ is the unique common transversal of r, s pasing through $\boldsymbol{o}$. The point $\boldsymbol{p}$ is thus in ψ. Hence the entire line l is contained in ψ as required.

It follows from this construction that ψ contains h and $\langle \boldsymbol{o}, \boldsymbol{b} \rangle$. To prove this, it suffices to note that $l = \Pi \cap \omega(\Pi)$ meets both h and $\langle \boldsymbol{o}, \boldsymbol{b} \rangle$. As Π varies, the points $l \cap h$ and $l \cap \langle \boldsymbol{o}, \boldsymbol{b} \rangle$ sweep out the whole of h and $\langle \boldsymbol{o}, \boldsymbol{b} \rangle$ respectively.

The degree of ψ is readily obtained. The surface ψ contains the skew lines h and $\langle \boldsymbol{o}, \boldsymbol{b} \rangle$, thus ψ is not a plane. The degree of ψ is thus at least two. Let k be a fixed line in space. As Π varies through the space of planes containing h, the linear transformation $\Pi \mapsto \omega(\Pi)$ induces a linear transformation ρ of k defined by $\rho(\Pi \cap k) = \omega(\Pi) \cap k$. The fixed points of ρ are precisely the points at which k intersects ψ. The projective linear transformation ρ has at most two fixed points, thus k meets ψ in at most two points. It follows that the degree of ψ is at most two. In consequence, the degree of ψ is exactly two.

2.3. *The intersection of complementary ambiguous surfaces*

It has been shown in §2.1 that each ambiguous case of the reconstruction problem has associated with it two complementary ambiguous surfaces, ψ and ϕ, defined by (7) and (8) respectively. The surfaces ψ, ϕ are each of degree two, thus the intersection $\psi \cap \phi$ is a space curve of degree four. Some properties of the curve $\psi \cap \phi$ are now obtained.

With the notation of §2.2, let $\boldsymbol{\Phi}$ be the plane $\langle h, \boldsymbol{o} \rangle$. It follows from the construction of ψ described in §2.2 that the line $g = \boldsymbol{\Phi} \cap \omega(\boldsymbol{\Phi})$ contains $\boldsymbol{o}$ and is contained in ψ. A comparison of (7) and (8) shows that the equation for ϕ is obtained from the equation for ψ by interchanging $\boldsymbol{a}$ and $\boldsymbol{b}$, and then replacing the orthogonal matrix U by U^{T}. It follows that ϕ is obtained by a construction similar to that of §2.2 by interchanging $\boldsymbol{a}$ and $\boldsymbol{b}$ and replacing ω by ω^{-1}. Thus ϕ is swept out by the lines $\Xi \cap \omega^{-1}(\Xi)$ as Ξ ranges through the pencil of planes containing $\omega\langle \boldsymbol{o}, \boldsymbol{a} \rangle$. In particular, $\omega(\boldsymbol{\Phi})$ contains $\omega\langle \boldsymbol{o}, \boldsymbol{a} \rangle$, thus ϕ contains the line

$$\omega(\boldsymbol{\Phi}) \cap \omega^{-1}\omega(\boldsymbol{\Phi}) = g.$$

It follows that the line g is contained in $\psi \cap \phi$, thus the space curve $\psi \cap \phi$ splits into g and a space curve c of degree $3 = 4-1$. The curve c is an example of a horopter curve (Buchanan 1987; Helmholtz 1925). It is the locus of points $\boldsymbol{x}$ such that

$$\boldsymbol{x} = \langle \boldsymbol{a}, \boldsymbol{x} \rangle \cap \omega\langle \boldsymbol{a}, \boldsymbol{x} \rangle. \tag{11}$$

The curve c meets every line of the form $l = \Pi \cap \omega(\Pi)$ twice as Π ranges through the pencil of planes containing h. This fact follows from the general properties of twisted cubics contained in quadric surfaces (Semple & Kneebone 1952). A direct proof is obtained by noting that if $\boldsymbol{p}$ is any point of l then $\omega\langle \boldsymbol{a}, \boldsymbol{p}\rangle$ meets l, and the correspondence $\boldsymbol{p} \mapsto \omega\langle \boldsymbol{a}, \boldsymbol{p}\rangle \cap l$ is a projective linear transformation from l to itself. A projective linear transformation of a line has, in general, exactly two fixed points. These two fixed points are precisely the points $l \cap c$. As a special case of this result, c meets the line g twice.

In contrast, c meets $\langle \boldsymbol{o}, \boldsymbol{b}\rangle$ exactly once. To show that c meets $\langle \boldsymbol{o}, \boldsymbol{b}\rangle$ note that $\omega\langle \boldsymbol{a}, \boldsymbol{b}\rangle$ contains $\boldsymbol{b}$. It follows that $\boldsymbol{b} = \langle \boldsymbol{a}, \boldsymbol{b}\rangle \cap \omega\langle \boldsymbol{a}, \boldsymbol{b}\rangle$ is a point of c. If c were to meet $\langle \boldsymbol{o}, \boldsymbol{b}\rangle$ twice then the plane spanned by $\langle \boldsymbol{o}, \boldsymbol{b}\rangle$ and any line $l = \Pi \cap \omega(\Pi)$ would intersect c at four points, contradicting the fact that c is a cubic space curve.

The above results can be obtained from (7) and (8) by algebraic arguments. For example, it follows from (7) and (8) that g is given in parametrized form by

$$t \mapsto t(\boldsymbol{a} \times U^{\mathrm{T}}\boldsymbol{b}) \times (U\boldsymbol{a} \times \boldsymbol{b}). \tag{12}$$

To prove that the line of (12) lies in ψ, it suffices to show that the vector

$$\boldsymbol{n} = (\boldsymbol{a} \times U^{\mathrm{T}}\boldsymbol{b}) \times (U\boldsymbol{a} \times \boldsymbol{b})$$

satisfies the equations

$$(U\boldsymbol{n} \times \boldsymbol{n}) \cdot \boldsymbol{b} = 0, \quad (U\boldsymbol{a} \times \boldsymbol{n}) \cdot \boldsymbol{b} = -(U\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{n} = 0.$$

The proof that g is contained in ϕ is similar. Full details are given by Negahdaripour (1990).

3. Projective geometry

The projective geometry relevant to the study of ambiguity is reviewed briefly. Projective geometry can be studied either synthetically, by using arguments based directly on the properties of geometric objects such as lines, planes, conics, etc., or analytically, by using polynomial equations (Semple & Kneebone 1952). The synthetic approach gives a greater insight into the underlying geometry, making it easier to formulate conjectures, and to find ways of proving or disproving them. In contrast, the analytic approach often leads to complicated polynomial equations, the properties of which are not immediately apparent. However, the analytic approach is easier to make rigorous, and it becomes essential when practical applications to computer vision are considered (Hofmann 1950). Both the synthetic and the analytic approaches are used as appropriate.

A point of projective n-dimensional space $\mathbb{P}^n$ is represented by an $(n+1)$-tuple of coordinates $(x_1, \ldots, x_{n+1})$, where at least one of the x_i is non-zero. Two $(n+1)$-tuples $(x_1, \ldots, x_{n+1})$ and $(y_1, \ldots, y_{n+1})$ represent the same point of $\mathbb{P}^n$ if and only if there exists a non-zero scalar λ such that $x_i = \lambda y_i$ for $1 \leqslant i \leqslant n+1$. The space $\mathbb{P}^1$ is called the projective line, and $\mathbb{P}^2$ is called the projective plane. A one-dimensional projective space is often referred to as a pencil, and a two-dimensional projective space is often referred to as a star. A projective subspace of $\mathbb{P}^n$ of dimension $n-1$ is called a hyperplane.

Let $\boldsymbol{p}$ be a point of $\mathbb{P}^n$ with coordinates $(x_1, \ldots, x_{n+1})$, and let x_j be any non-zero coordinate of $\boldsymbol{p}$. Then $\boldsymbol{p}$ is also represented by the coordinates $(x_1/x_j, \ldots, x_{n+1}/x_j)$, with 1 in the jth position. The coordinates $(x_1/x_j, \ldots, x_{n+1}/x_j)$ are inhomogeneous coordinates of $\boldsymbol{p}$, and the original coordinates $(x_1, \ldots, x_{n+1})$ are homogeneous coordinates of $\boldsymbol{p}$. The values of the inhomogeneous coordinates of $\boldsymbol{p}$ depend on the choice of j. In the case of $\mathbb{P}^1$, a point with homogeneous coordinates (x_1, x_2) has inhomogeneous coordinates $(\alpha, 1)$ where $\alpha = x_1/x_2$. The points of $\mathbb{P}^1$ are thus parametrized by a single scalar variable α, and every point of $\mathbb{P}^1$ is included in the parametrization provided the value $\alpha = \infty$, corresponding to (1,0) is allowed.

The effectiveness of projective geometry is partly due to the fact that projective spaces arise naturally in many different circumstances. For example, in $\mathbb{P}^2$ the lines containing a fixed point form a pencil of lines, in $\mathbb{P}^3$ the lines containing a fixed point form a star of lines, and again in $\mathbb{P}^3$, the planes containing a fixed line form a pencil of planes. Stars of lines arise in §2.2 as part of a model for the process of image formation, and pencils of planes arise in §2.2 in the geometric construction of ambiguous surfaces.

3.1. *Collineations*

A collineation is an invertible linear transformation from one projective space to another. If coordinate systems are chosen for the two projective spaces then the collineation is represented by an invertible matrix. Two invertible matrices represent the same collineation if and only if one matrix is a non-zero scalar multiple of the other.

A special notation is employed for collineations of $\mathbb{P}^1$. Let ω be a collineation of $\mathbb{P}^1$, and let $\boldsymbol{x}, \boldsymbol{y}$ be points of $\mathbb{P}^1$ such that $\omega \boldsymbol{x} = \boldsymbol{y}$. The correspondence $\boldsymbol{x} \leftrightarrow \boldsymbol{y}$ is called a homography, and is denoted by $\boldsymbol{x} \barwedge \boldsymbol{y}$. If $\boldsymbol{x} \barwedge \boldsymbol{y}$ implies $\boldsymbol{y} \barwedge \boldsymbol{x}$ then the homography and the associated collineation are both referred to as involutions. A collineation is an involution if and only if it is equal to its own inverse.

Let $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_{n+2}$ be points of $\mathbb{P}^n$ chosen such that no hyperplane of $\mathbb{P}^n$ contains $n+1$ of the $\boldsymbol{x}_i$. Then any collineation ω from $\mathbb{P}^n$ to a second projective space is uniquely determined by the $n+2$ points $\omega \boldsymbol{x}_i$. Conversely, if any $n+2$ points $\boldsymbol{y}_i$ are selected in a second projective space $\mathbb{P}^{n'}$ then there exists a unique collineation ω from $\mathbb{P}^n$ to $\mathbb{P}^{n'}$ such that $\omega \boldsymbol{x}_i = \boldsymbol{y}_i$. In particular, if $\boldsymbol{x}_i = \boldsymbol{y}_i$ for all i, then ω is the identity. As a special case, a collineation of $\mathbb{P}^1$ is uniquely defined by its values at three distinct points, and any collineation from $\mathbb{P}^1$ to itself that fixes three distinct points is the identity.

3.2. *Conics*

A conic s is a set of points of $\mathbb{P}^2$ satisfying an equation $\boldsymbol{x}^{\mathrm{T}} M \boldsymbol{x} = 0$, where M is a symmetric 3×3 matrix. Two symmetric 3×3 matrices represent the same conic if and only if one matrix is a non-zero scalar multiple of the other. The conic s is non-singular if and only if M has a non-zero determinant.

Each non-singular conic defines a collineation, known as a polarity, from the points of $\mathbb{P}^2$ to the lines of $\mathbb{P}^2$. To define the polarity π associated with s, let $\boldsymbol{a}$ be any point of $\mathbb{P}^2$, and let the two tangents to s drawn from $\boldsymbol{a}$ touch s at $\boldsymbol{p}, \boldsymbol{q}$ as shown in figure 4. Then $\pi \boldsymbol{a} = \langle \boldsymbol{p}, \boldsymbol{q} \rangle$. If $\boldsymbol{a}$ is a point of s then $\pi \boldsymbol{a}$ is the tangent line to s at $\boldsymbol{a}$. The line $\pi \boldsymbol{a}$ is said to be the polar of $\boldsymbol{a}$ with respect to s, and $\boldsymbol{a}$ is said to be the pole of $\pi \boldsymbol{a}$ with respect to s.

The transformation $\boldsymbol{a} \mapsto \pi \boldsymbol{a}$ is given in coordinate form by $\boldsymbol{a} \mapsto \boldsymbol{a}^{\mathrm{T}} M$, since a point $\boldsymbol{x}$ of $\mathbb{P}^2$ is on the line $\pi \boldsymbol{a}$ if and only if $\boldsymbol{a}^{\mathrm{T}} M \boldsymbol{x} = 0$. This shows that the transformation $\boldsymbol{a} \mapsto \pi \boldsymbol{a}$ is a collineation as claimed.

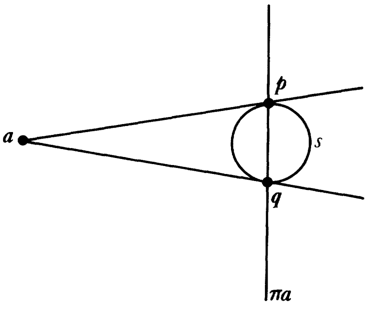

Figure 4. The polar of a point with respect to a conic.

3.3. *Euclidean transformations and the absolute conic*

The problem of reconstructing a scene from image correspondences involves both projection, as a model for the process of image formation, and rigidity, as a basic assumption about the scene in view. Thus the formulation of the reconstruction problem within projective geometry requires a projective geometric treatment of rigidity. This is done by selecting a subset of the collineations of $\mathbb{P}^3$ known as the euclidean transformations. Euclidean transformations are defined to be collineations from $\mathbb{P}^3$ to itself that leave invariant a particular conic known as the absolute conic. Euclidean transformations preserve angles and the ratios of lengths, thus they preserve the properties of rigid configurations of points up to a uniform change of scale. This scale change does not add to the difficulty of the reconstruction problem because reconstruction from image correspondences always involves an unknown scale factor.

Let an arbitrary system of coordinates be chosen for $\mathbb{P}^3$. The absolute conic, Ω, is a non-singular conic without real points, chosen in $\mathbb{P}^3$. The plane containing Ω is referred to as the plane at infinity, Π_∞. It is immaterial which conic is chosen as the absolute conic, because the same theory of euclidean transformations is obtained independently of the choice. It is, however, important to keep Ω fixed once the choice has been made. The euclidean transformations are, by definition, those collineations of $\mathbb{P}^3$ that leave Ω invariant.

The close connection between euclidean transformations and rigid displacements becomes apparent on working out the consequences of a particular choice for Ω. Let (x_1, x_2, x_3, x_4) be the coordinates of $\mathbb{P}^3$ and let Ω be the conic

$$x_4 = 0, \quad x_1^2 + x_2^2 + x_3^2 = 0. \tag{13}$$

The plane Π_∞ is $x_4 = 0$. Let ρ be a rigid displacement of $\mathbb{R}^3$. Then ρ has the form

$$[x_1, x_2, x_3] \mapsto [x_1 + t_1, x_2 + t_2, x_3 + t_3] R^{\mathrm{T}}, \tag{14}$$

where R is an orthogonal matrix, and $\boldsymbol{t}$ is a translation vector. (Equation (14) is expressed in terms of row vectors only because this simplifies the layout of the formulae on the page.) The extension of ρ to $\mathbb{P}^3$ is given by

$$(x_1, x_2, x_3, x_4) \mapsto ([x_1 + x_4 t_1, x_2 + x_4 t_2, x_3 + x_4 t_3] R^{\mathrm{T}}, x_4).$$

The restriction of ρ to Π_∞ is given by

$$(x_1, x_2, x_3, 0) \mapsto ([x_1, x_2, x_3] R^{\mathrm{T}}, 0). \tag{15}$$

It follows from (13) and (15) that $\rho(\Omega) = \Omega$, thus the rigid displacement ρ is a euclidean transformation. The converse result holds up to a scale factor, as shown in the following theorem.

Theorem 3.1. *Let a cartesian coordinate system be chosen for* $\mathbb{P}^3$. *Then a collineation* ω *of* $\mathbb{P}^3$ *is a euclidean transformation if and only if*

$$\omega = \begin{bmatrix} r_{11} & r_{12} & r_{13} & t_1 \\ r_{21} & r_{22} & r_{23} & t_2 \\ r_{31} & r_{32} & r_{33} & t_3 \\ 0 & 0 & 0 & t_4 \end{bmatrix}, \tag{16}$$

where $t_4 \neq 0$ *and the* 3×3 *matrix* R *formed by the* r_{ij} *satisfies* $R^{\mathrm{T}}R = \mu I$ *for some non-zero scalar* μ.

Proof. If ω is given by (16) then $\omega(\Omega) = \Omega$, thus ω is a euclidean transformation. Conversely, let ω be a euclidean transformation of $\mathbb{P}^3$ with the matrix representation

$$\omega = \begin{bmatrix} r_{11} & r_{12} & r_{13} & t_1 \\ r_{21} & r_{22} & r_{23} & t_2 \\ r_{31} & r_{32} & r_{33} & t_3 \\ r_{41} & r_{42} & r_{43} & t_4 \end{bmatrix}.$$

Define $\boldsymbol{r}_1$, $\boldsymbol{r}_2$, $\boldsymbol{r}_3$ by $\boldsymbol{r}_1 = (r_{11}, r_{12}, r_{13})^{\mathrm{T}}$, $\boldsymbol{r}_2 = (r_{21}, r_{22}, r_{23})^{\mathrm{T}}$, $\boldsymbol{r}_3 = (r_{31}, r_{32}, r_{33})^{\mathrm{T}}$. The absolute conic Ω contains the points $\boldsymbol{p} = (1, \mathrm{i}, 0, 0)^{\mathrm{T}}$, $\boldsymbol{q} = (1, -\mathrm{i}, 0, 0)^{\mathrm{T}}$, and by hypothesis, $\omega(\Omega) = \Omega$, thus

$$\omega\boldsymbol{p}\cdot\omega\boldsymbol{p} = \omega\boldsymbol{q}\cdot\omega\boldsymbol{q} = 0, \quad (\omega\boldsymbol{p})_4 = (\omega\boldsymbol{q})_4 = 0.$$

It follows that

$$(\boldsymbol{r}_1 + \mathrm{i}\boldsymbol{r}_2)\cdot(\boldsymbol{r}_1 + \mathrm{i}\boldsymbol{r}_2) = (\boldsymbol{r}_1 - \mathrm{i}\boldsymbol{r}_2)\cdot(\boldsymbol{r}_1 - \mathrm{i}\boldsymbol{r}_2) = 0,$$

$$r_{41} + \mathrm{i}r_{42} = r_{41} - \mathrm{i}r_{42} = 0,$$

hence $\boldsymbol{r}_1\cdot\boldsymbol{r}_1 = \boldsymbol{r}_2\cdot\boldsymbol{r}_2$, $\boldsymbol{r}_1\cdot\boldsymbol{r}_2 = 0$, and $r_{41} = r_{42} = 0$. Similarly, Ω contains the points $(0, 1, \mathrm{i}, 0)^{\mathrm{T}}$, $(0, 1, -\mathrm{i}, 0)^{\mathrm{T}}$, thus $\boldsymbol{r}_2\cdot\boldsymbol{r}_2 = \boldsymbol{r}_3\cdot\boldsymbol{r}_3$, $\boldsymbol{r}_2\cdot\boldsymbol{r}_3 = 0$ and $r_{43} = 0$. The result follows on setting

$$\mu = \boldsymbol{r}_1\cdot\boldsymbol{r}_1 = \boldsymbol{r}_2\cdot\boldsymbol{r}_2 = \boldsymbol{r}_3\cdot\boldsymbol{r}_3.$$

The variables μ and t_4 are non-zero since ω is a collineation and thus, by definition, invertible. □

A coordinate system in which Ω has the form (13) is referred to as a cartesian coordinate system. The origin of cartesian coordinates is usually chosen to be the point $(0, 0, 0, 1)^{\mathrm{T}}$. The space $\mathbb{P}^3 \backslash \Pi_\infty$ is the euclidean space $\mathbb{R}^3$. The points of $\mathbb{R}^3$ (as a subset of $\mathbb{P}^3$) have inhomogeneous coordinates $(x_1, x_2, x_3, 1)$, which yield the usual coordinates $[x_1, x_2, x_3]$ within $\mathbb{R}^3$.

Once Ω is chosen, orthogonality of vectors is defined with respect to Ω. Two points $\boldsymbol{m}, \boldsymbol{n}$ of Π_∞ are said to be orthogonal if $\boldsymbol{m}$ is on the polar line of $\boldsymbol{n}$ with respect to Ω. In cartesian coordinates this definition of orthogonality reduces to the usual definition, $\boldsymbol{m}\cdot\boldsymbol{n} = 0$.

It follows from theorem 3.1 that in a cartesian coordinate system a collineation of Π_∞ leaves Ω invariant if and only if it is represented by an orthogonal matrix. In consequence, the properties of orthogonal matrices, and hence the properties of rotations are 'really' properties of conics. Theorem 3.2 illustrates this point of view. An orthogonal collineation is, by definition, a collineation of Π_∞ that leaves Ω invariant.

Theorem 3.2. *Let ω be an orthogonal collineation of the plane at infinity such that ω^2 is not the identity, and let $\boldsymbol{n}$ be a fixed point of ω not contained in the absolute conic. Then the polar of $\boldsymbol{n}$ with respect to the absolute conic intersects the absolute conic at fixed points of ω.*

Proof. Let $\pi\boldsymbol{n}$ be the polar of $\boldsymbol{n}$ with respect to the absolute conic Ω, and let $\boldsymbol{i}_n, \boldsymbol{j}_n$ be the two points of $\pi\boldsymbol{n} \cap \Omega$. From the definition of $\pi\boldsymbol{n}$ given in §3.2, $\boldsymbol{i}_n$ and $\boldsymbol{j}_n$ are the points of contact of the tangents drawn from $\boldsymbol{n}$ to Ω. Both Ω and $\pi\boldsymbol{n}$ are invariant under ω, thus $\boldsymbol{i}_n, \boldsymbol{j}_n$ are either fixed by ω or interchanged by ω.

If $\boldsymbol{i}_n$, $\boldsymbol{j}_n$ are fixed by ω then the result follows. Suppose, if possible, that ω interchanges $\boldsymbol{i}_n$ and $\boldsymbol{j}_n$. The line $\langle \boldsymbol{i}_n, \boldsymbol{j}_n \rangle$ is invariant under ω, thus ω has a fixed point m on $\langle \boldsymbol{i}_n, \boldsymbol{j}_n \rangle$. Let $\boldsymbol{i}_m, \boldsymbol{j}_m$ be the points of contact of the tangents drawn from $\boldsymbol{m}$ to Ω. Then ω^2 fixes the four points $\boldsymbol{i}_m, \boldsymbol{j}_m, \boldsymbol{i}_n, \boldsymbol{j}_n$. All four points are on Ω, thus no three of the points are collinear. It follows that ω^2 is the identity, contrary to hypothesis. □

Corollary. *The point $\boldsymbol{n}$ is the unique fixed point of ω not contained in Ω. To show this, it suffices to note that ω has an additional fixed point $\boldsymbol{m}$ not contained in Ω only if ω^2 is the identity.*

The reference in theorem 3.2 to $\boldsymbol{i}_n$ and $\boldsymbol{j}_n$ as fixed points of ω is at first sight confusing. If A is the matrix representing ω then

$$A\boldsymbol{i}_n = \lambda \boldsymbol{i}_n, \quad A\boldsymbol{j}_n = \mu \boldsymbol{j}_n,$$

where λ, μ are eigenvalues of A not equal to 1. However, in projective geometry, $\boldsymbol{i}_n$ is identified with $\lambda \boldsymbol{i}_n$, and $\boldsymbol{j}_n$ is identified with $\mu \boldsymbol{j}_n$, even though $\lambda, \mu \neq 1$. It is thus legitimate to write

$$\omega \boldsymbol{i}_n = \boldsymbol{i}_n, \quad \omega \boldsymbol{j}_n = \boldsymbol{j}_n,$$

and to refer to $\boldsymbol{i}_n$ and $\boldsymbol{j}_n$ as fixed points of ω.

3.4. *Quadric surfaces*

A quadric ψ is a set of points in $\mathbb{P}^3$ satisfying an equation of the form $\boldsymbol{x}^{\mathrm{T}} M \boldsymbol{x} = 0$, where M is a symmetric 4×4 matrix. The quadric ψ is non-singular if and only if M has a non-zero determinant. A straight line contained in ψ is known as a generator of ψ. The generators of ψ are extremely useful for describing the geometry of ψ.

The generators of ψ form two disjoint one-parameter families, $\mathscr{F}_1$ and $\mathscr{F}_2$. Each point $\boldsymbol{p}$ of ψ is contained in exactly two generators, $g_1(\boldsymbol{p})$ and $g_2(\boldsymbol{p})$ belonging to $\mathscr{F}_1$ and $\mathscr{F}_2$ respectively. Two generators of ψ intersect if and only if they belong to different families. Thus a fixed generator $g_1(\boldsymbol{p})$ of $\mathscr{F}_1$ intersects every generator of $\mathscr{F}_2$, and a fixed generator $g_2(\boldsymbol{p})$ of $\mathscr{F}_2$ intersects every generator of $\mathscr{F}_1$. These properties of generators are illustrated in figure 5.

An algebraic curve contained in ψ is said to be a (m, n) curve if it has m intersections with each generator of $\mathscr{F}_1$, and n intersections with each generator of $\mathscr{F}_2$. For example, the generators of $\mathscr{F}_1$ are $(0, 1)$ curves, and the generators of $\mathscr{F}_2$ are $(1, 0)$ curves. The conics contained in ψ are $(1, 1)$ curves.

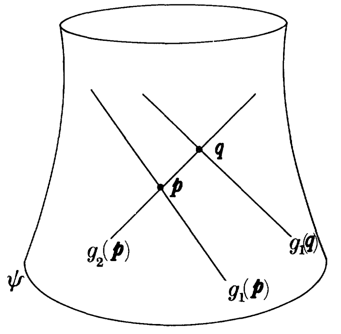

Figure 5. Generators of a quadric surface.

The intersection of two non-singular quadrics ψ_1, ψ_2 is, in general, a space curve of degree four which meets each generator of ψ_1 and ψ_2 twice. If ψ_1 and ψ_2 have a common generator g, then $\psi_1 \cap \psi_2 = g \cup c$, where c is a space curve of degree three meeting g twice. The curve c is either a $(2, 1)$ curve or a $(1, 2)$ curve.

The ambiguous surfaces described in §2 are, in general, non-singular quadrics, thus the geometry of quadrics is applicable to the study of ambiguity. Ambiguous surfaces contain generators with real (as opposed to complex) points, thus ambiguous surfaces are hyperboloids of one sheet (Longuet-Higgins 1988). Conversely, each hyperboloid of one sheet is an ambiguous surface.

3.5. *Twisted cubics*

A twisted cubic, c, is a curve of $\mathbb{P}^3$ parametrized by cubic polynomials in a single variable. Thus, by definition, there exists a transformation

$$(1, t) \mapsto (f(t), g(t), h(t), k(t))$$

from $\mathbb{P}^1$ to c, where $f(t), g(t), h(t), k(t)$ are polynomials in t of degree three or less. To avoid degenerate cases it is assumed that c is not entirely contained in a single plane. Twisted cubics have already been encountered in §2.3 as one component of the space curve formed by the intersection of two complementary ambiguous surfaces.

Any two non-singular twisted cubics are related by a collineation. For example, if c_1, c_2 are twisted cubics with respective parametrizations

$$t \mapsto A_1(1, t, t^2, t^3)^{\mathrm{T}}, \quad t \mapsto A_2(1, t, t^2, t^3)^{\mathrm{T}},$$

where A_1, A_2 are invertible 4×4 matrices, then $\omega(c_1) = c_2$, where ω is the collineation of $\mathbb{P}^3$ defined by $\omega = A_2 A_1^{-1}$.

If c is projected from a point $\boldsymbol{p}$ on c then the resulting curve, s, is a conic. This is proved as follows. An arbitrary plane through $\boldsymbol{p}$ meets c at $\boldsymbol{p}$ and also at two further points. The plane projects to a line meeting s at two points, thus s is of degree two. Hence s is a conic.

There are many different ways of characterizing twisted cubics apart from parametrization by cubic polynomials. A particularly useful way is star generation (Semple & Kneebone 1952), which is illustrated in figure 6. In the figure, the points $\boldsymbol{a}$ and $\boldsymbol{b}$ are centres of stars of lines in $\mathbb{P}^3$, and ω is a collineation from the star of lines through $\boldsymbol{a}$ to the star of lines through $\boldsymbol{b}$. A point $\boldsymbol{x}$ is on the twisted cubic c if and only if

$$\omega\langle \boldsymbol{a}, \boldsymbol{x}\rangle = \langle \boldsymbol{b}, \boldsymbol{x}\rangle. \tag{17}$$

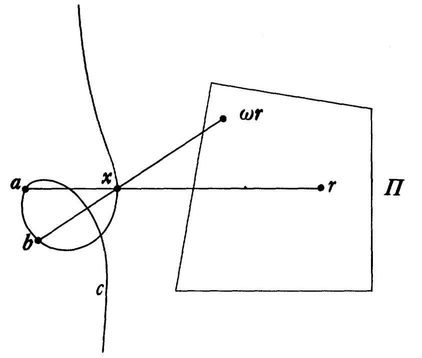

Figure 6. Star generation of a twisted cubic.

It can be shown that c uniquely determines ω. Conversely, if ω is given, then the locus of points $\boldsymbol{x}$ satisfying (17) is a twisted cubic.

It is often convenient to parametrize the stars of lines centred at $\boldsymbol{a}$ and $\boldsymbol{b}$ by taking the intersections of each line with a fixed plane Π. In this case, ω can be regarded as a collineation of Π. A point $\boldsymbol{x}$ is on c if and only if there is a point $\boldsymbol{r}$ of Π such that

$$\boldsymbol{x} = \langle \boldsymbol{a}, \boldsymbol{r} \rangle \cap \langle \boldsymbol{b}, \omega \boldsymbol{r} \rangle. \tag{18}$$

In the star generation of c, $\boldsymbol{a}$ and $\boldsymbol{b}$ can be any two distinct points of c. Different choices of $\boldsymbol{a}$ and $\boldsymbol{b}$ yield different collineations ω.

4. Involutions

An involution is a collineation τ from $\mathbb{P}^n$ to itself such that τ^2 is the identity. If τ is the identity then it is said to be the trivial involution. Involutions are particularly simple examples of collineations. In spite of this simplicity they play an important role in the geometry of ambiguity. For example, ambiguous surfaces and horopter curves are both invariant under non-trivial rigid involutions.

Involutions of $\mathbb{P}^n$, involutions of conics, rigid involutions, and involutions of twisted cubics are described, with particular emphasis on the fixed points of the involutions. It turns out that the various types of involution are closely interrelated. For example, rigid involutions are involutions of the absolute conic, and there are close connections between involutions of conics, involutions of twisted cubics and involutions of $\mathbb{P}^1$.

4.1. *Involutions of* $\mathbb{P}^n$

The general form of an involution of $\mathbb{P}^n$ is obtained. In reconstruction the cases of most interest are those for which $n \leqslant 3$.

Theorem 4.1. *Let τ be an involution of $\mathbb{P}^n$. Then coordinates can be chosen such that τ is represented by a diagonal matrix with each non-zero entry equal to ± 1, and such that the positive entries precede the negative entries on the diagonal.*

Proof. The result is obtained by induction starting at $n = 1$. Let τ be an involution of $\mathbb{P}^1$, and let $\boldsymbol{b}$ be a fixed point of τ. If τ is the identity then the result holds. If τ is not the identity then there exists a point $\boldsymbol{a}$ of $\mathbb{P}^1$ such that $\tau\boldsymbol{a} \neq \boldsymbol{a}$. Let coordinates of $\mathbb{P}^1$ be chosen such that $\boldsymbol{a} = (1, 1)^{\mathrm{T}}$, $\tau\boldsymbol{a} = (1, -1)^{\mathrm{T}}$, and $\boldsymbol{b} = (1, 0)^{\mathrm{T}}$. Then τ is represented by a matrix of the required form.

Now suppose $n > 1$. The involution τ induces an involution τ' of the n dimensional space of hyperplanes of $\mathbb{P}^n$. If H is a hyperplane of $\mathbb{P}^n$, then $\tau' H$ is the set of points $\tau \boldsymbol{x}$ as $\boldsymbol{x}$ varies over the points of H. The involution τ' has at least one fixed point, thus there exists a hyperplane H such that $\tau' H = H$. It follows that H is invariant under τ.

The hyperplane H has dimension $n-1$, thus by the induction hypothesis, a basis $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$ of H can be found such that the restriction of τ to H is represented by a diagonal matrix with non-zero entries equal to ± 1. The basis $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$ of H is extended to a basis $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_{n+1}$ of $\mathbb{P}^n$ by choosing $\boldsymbol{x}_{n+1}$ to be a fixed point of τ outside H. A suitable point $\boldsymbol{x}_{n+1}$ always exists for the following reason. Let $\boldsymbol{a}$ be any point of $\mathbb{P}^n$ not contained in H. If $\tau \boldsymbol{a} = \boldsymbol{a}$ then set $\boldsymbol{x}_{n+1} = \boldsymbol{a}$. If $\tau \boldsymbol{a} \neq \boldsymbol{a}$ then the line $\langle \boldsymbol{a}, \tau \boldsymbol{a} \rangle$ is invariant under τ. It follows from the case $n = 1$ that τ has two fixed points on $\langle \boldsymbol{a}, \tau \boldsymbol{a} \rangle$. One of these fixed points is in H and one is outside H. This second fixed point is chosen to be $\boldsymbol{x}_{n+1}$.

By construction, the matrix A representing τ in the basis $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_{n+1}$ is diagonal. As τ^2 is the identity, there exists a non-zero scalar λ such that $A_{ii} = \pm \lambda$ for $1 \leqslant i \leqslant n+1$. The matrix A is determined by τ only up to a non-zero scale factor, thus without loss of generality $\lambda = 1$. The basis $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_{n+1}$ is reordered if necessary to ensure that the entries $A_{ii} = 1$ precede the entries $A_{ii} = -1$ on the leading diagonal of A. □

A complete description of the involutions of $\mathbb{P}^1$, $\mathbb{P}^2$, $\mathbb{P}^3$ is obtained from theorem 4.1.

Theorem 4.2. *A non-trivial involution of $\mathbb{P}^1$ has exactly two fixed points. A non-trivial involution of $\mathbb{P}^2$ has a line of fixed points and a single fixed point not on the line. A non-trivial involution of $\mathbb{P}^3$ has either (a) two skew lines of fixed points; or (b) a plane of fixed points and a single fixed point not contained in the plane. If an involution of $\mathbb{P}^1$, $\mathbb{P}^2$, or $\mathbb{P}^3$ has strictly more fixed points than those listed then the involution is the identity.*

Proof. A proof is given for $\mathbb{P}^3$. The proofs for $\mathbb{P}^1$ and $\mathbb{P}^2$ are similar. Let τ be a non-trivial involution of $\mathbb{P}^3$. It follows from theorem 4.1 that a basis of $\mathbb{P}^3$ can be found in which τ is represented by a matrix A of the form

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \tag{19}$$

or by a matrix B of the form

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}. \tag{20}$$

In the first case τ has two skew lines of fixed points, namely, $t \mapsto (t, 1, 0, 0)^{\mathrm{T}}$ and $t \mapsto (0, 0, t, 1)^{\mathrm{T}}$. In the second case τ has a plane of fixed points $x_1 = 0$, and a fixed point $(1, 0, 0, 0)^{\mathrm{T}}$ not in the plane. It follows from (19) and (20) that τ has no additional fixed points beyond those specified in the statement of the theorem. □

The isolated fixed point of an involution τ of $\mathbb{P}^2$ is known as the vertex of τ, and denoted by $\boldsymbol{p}_\tau$. If τ is a rotation through 180° then $\boldsymbol{p}_\tau$ may also be referred to as the axis of τ. The involutions of $\mathbb{P}^3$ with two skew lines of fixed points are known as skew involutions, and the involutions of $\mathbb{P}^3$ with a plane of fixed points and one additional fixed point are known as harmonic homologies (Semple & Kneebone 1952). The rotations of $\mathbb{P}^3$ through 180° are examples of skew involutions, and the reflections of $\mathbb{P}^3$ in a plane are examples of harmonic homologies.

Theorem 4.3. *An involution of $\mathbb{P}^n$ is uniquely determined by its fixed points.*

Proof. The result is immediate if the involution is the identity. Suppose then that τ is a non-trivial involution of $\mathbb{P}^1$. It follows from theorem 4.2 that τ has exactly two fixed points $\boldsymbol{p}$, $\boldsymbol{q}$ in $\mathbb{P}^1$. Let coordinates of $\mathbb{P}^1$ be chosen such that $\boldsymbol{p} = (1, 0)^{\mathrm{T}}$, $\boldsymbol{q} = (0, 1)^{\mathrm{T}}$. Then τ is represented by a diagonal 2×2 matrix with entries $1, -1$, down the leading diagonal. Thus τ is uniquely determined by $\boldsymbol{p}$, $\boldsymbol{q}$.

Next, let τ be a non-trivial involution of $\mathbb{P}^n$, where $n > 1$. Let $\boldsymbol{a}$ be a point of $\mathbb{P}^n$ such that $\boldsymbol{a} \neq \tau\boldsymbol{a}$. The line $\langle \boldsymbol{a}, \tau\boldsymbol{a}\rangle$ is invariant under τ. It follows from the result for $\mathbb{P}^1$ that the restriction of τ to $\langle \boldsymbol{a}, \tau\boldsymbol{a}\rangle$ is determined by the fixed points of τ contained in $\langle \boldsymbol{a}, \tau\boldsymbol{a}\rangle$. □

Theorem 4.4. *Any line containing the vertex of a non-trivial involution of $\mathbb{P}^2$ is invariant under the involution, and conversely, any line invariant under the involution either contains the vertex of the involution or it coincides with the line of fixed points of the involution.*

Proof. Let τ be a non-trivial involution of $\mathbb{P}^2$ with vertex $\boldsymbol{p}_\tau$ and line of fixed points g_τ. Let k be any line containing $\boldsymbol{p}_\tau$. Then k contains two fixed points of τ, namely, $\boldsymbol{p}_\tau$ and $k \cap g_\tau$, thus k is invariant under τ. Conversely, let k be any line of $\mathbb{P}^2$ invariant under τ. If k contains three or more fixed points of τ then every point of k is a fixed point of τ and thus $k = g_\tau$. If k contains exactly two fixed points of τ, then at least one of these points does not lie on g_τ, thus one of the fixed points of τ in k is $\boldsymbol{p}_\tau$. It follows from theorem 4.2 that k does not contain strictly less than two fixed points of τ. □

4.2. *Involutions of conics*

An involution of a conic s is defined to be an involution of the plane containing s that leaves s invariant. There are close connections between involutions of conics and involutions of $\mathbb{P}^1$, because a conic has a natural structure as a one-dimensional projective space (Semple & Kneebone 1952). The next theorem illustrates this latter connection, in that the result is similar in form to the part of theorem 4.2 applying to $\mathbb{P}^1$.

Theorem 4.5. *A non-trivial involution of a non-singular conic has exactly two fixed points on the conic, and these two fixed points uniquely determine the involution.*

Proof. Let τ be a non-trivial involution of a non-singular conic s, let $\boldsymbol{p}_\tau$ be the vertex of τ and let g_τ be the line of fixed points of τ. Suppose, if possible, that $\boldsymbol{p}_\tau$ is on s. If $\boldsymbol{p}$ is any point of s then by theorem 4.4, $\langle \boldsymbol{p}_\tau, \boldsymbol{p}\rangle$ is invariant under τ. It follows that $\langle \boldsymbol{p}_\tau, \boldsymbol{p}\rangle \cap s$ is invariant under τ, hence $\tau\boldsymbol{p} = \boldsymbol{p}$. It follows that the restriction of τ to s is the identity. Hence τ is the identity, contrary to hypothesis. The point $\boldsymbol{p}_\tau$ is thus not on s.

As $\boldsymbol{p}_\tau$ is not on s there exist two tangent lines from $\boldsymbol{p}_\tau$ to s. These tangents are invariant under τ, thus the points $\boldsymbol{r}, \boldsymbol{s}$ at which the two tangents touch s are fixed points of τ (see figure 7). Now τ has at most two fixed points on s, namely the points

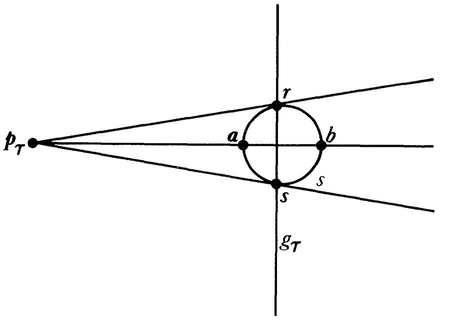

Figure 7. Involutions of conics.

of $g_\tau \cap s$. Thus τ has exactly two fixed points on s. It follows that $g = \langle \boldsymbol{r}, \boldsymbol{s} \rangle$. The proof is completed by noting that $\boldsymbol{r}, \boldsymbol{s}$ uniquely determine g_τ, $\boldsymbol{p}_\tau$, thus $\boldsymbol{r}, \boldsymbol{s}$ uniquely determine τ. □

Corollary. *Two distinct points $\boldsymbol{a}, \boldsymbol{b}$ of s are interchanged by τ if and only if $\langle \boldsymbol{a}, \boldsymbol{b} \rangle$ contains $\boldsymbol{p}_\tau$. The corollary follows from theorem* 4.4 *and the fact that $\boldsymbol{r}, \boldsymbol{s}$ are the only fixed points of τ on s.*

In the next theorem on the existence of involutions of conics relies on Steiner's theorem (Semple & Kneebone 1952). Steiner's theorem states that if $\boldsymbol{a}, \boldsymbol{b}$ are two distinct points of $\mathbb{P}^2$ and if ω is a correspondence from the pencil of lines k through $\boldsymbol{a}$ to the pencil of lines through $\boldsymbol{b}$ such that $k \barwedge \omega k$ then the locus $k \cap \omega k$ is a conic. Conversely, if s is a conic, $\boldsymbol{a}, \boldsymbol{b}$ are any two distinct points of s, and $\boldsymbol{x}$ is a variable point of s, then the correspondence $\langle \boldsymbol{a}, \boldsymbol{x} \rangle \leftrightarrow \langle \boldsymbol{b}, \boldsymbol{x} \rangle$ establishes a homography between the pencil of lines through $\boldsymbol{a}$ and the pencil of lines through $\boldsymbol{b}$.

Theorem 4.6. *Let two distinct points be given on a non-singular conic. Then there is a unique non-trivial involution of the conic that fixes the two points.*

Proof. Let $\boldsymbol{r}, \boldsymbol{s}$ be distinct points on a non-singular conic s. It follows from Steiner's theorem that there exists a collineation ω from the pencil of lines through $\boldsymbol{r}$ to the pencil of lines through $\boldsymbol{s}$ such that s is the locus of the intersections $k \cap \omega k$ as k varies through the pencil of lines containing $\boldsymbol{r}$. It also follows from Steiner's theorem that if g, h are the tangents to s at $\boldsymbol{r}, \boldsymbol{s}$ respectively, then $\omega g = \langle \boldsymbol{r}, \boldsymbol{s} \rangle$, $\omega \langle \boldsymbol{r}, \boldsymbol{s} \rangle = h$.

Let τ be the unique involution of $\mathbb{P}^2$ with a line of fixed points $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$, and a vertex $\boldsymbol{p}_\tau$ taken to be the pole of $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$ with respect to s. The points $\boldsymbol{r}, \boldsymbol{s}$ are fixed by τ, thus τ induces involutions τ_r, τ_s, respectively, of the pencils of lines through $\boldsymbol{r}$ and $\boldsymbol{s}$. Now $\omega^{-1} \tau_s \omega$ is an involution of the pencil of lines through $\boldsymbol{r}$ with the same fixed lines as τ_r, namely $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$ and $\langle \boldsymbol{r}, \boldsymbol{p}_\tau \rangle$, thus by theorem 4.3, $\tau_r = \omega^{-1} \tau_s \omega$.

The conic s is the locus of $k \cap \omega k$ as k varies through the pencil of lines containing $\boldsymbol{r}$, thus the conic $\tau(s)$ is the locus of

$$\tau(k \cap \omega k) = \tau_r k \cap \tau_s \omega k = \tau_r k \cap \omega \tau_r k. \tag{21}$$

As k varies, $\tau_r k$ varies through the pencil of lines containing $\boldsymbol{r}$. It thus follows from (21) that $\tau(s) = s$. Hence τ is an involution of s with fixed points $\boldsymbol{r}, \boldsymbol{s}$. The uniqueness of τ and the fact that $\boldsymbol{r}, \boldsymbol{s}$ are the only fixed points of τ on s follow from theorem 4.5. □

Theorem 4.7. *Any point in the plane containing a non-singular conic, but not on the conic, is the vertex of a unique non-trivial involution of the conic.*

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Proof. Let s be a non-singular conic, let $\boldsymbol{v}$ be a point in the plane containing s but not on s, and let $\boldsymbol{r}, \boldsymbol{s}$ be the points of contact of the tangents drawn from $\boldsymbol{v}$ to s. It follows from theorem 4.6 that there is a unique involution τ of s with fixed points $\boldsymbol{r}$, $\boldsymbol{s}$ on s. It follows from the proof of theorem 4.5 that the vertex of τ is equal to $\boldsymbol{v}$. The points $\boldsymbol{r}, \boldsymbol{s}$ are uniquely determined by $\boldsymbol{v}$, thus τ is uniquely determined by $\boldsymbol{v}$. □

An example of an involution of a conic is given. Let s be the unit circle in $\mathbb{R}^3$ lying in the x, y plane with centre at the origin. A non-trivial involution τ of s is obtained by rotating s through 180° about the x axis. The vertex of τ is the point at infinity on the y axis. The involution τ interchanges two points $\boldsymbol{a}, \boldsymbol{b}$ of s if and only if $\langle \boldsymbol{a}, \boldsymbol{b} \rangle$ is parallel to the y axis.

4.3. *Orthogonal involutions of Π_∞ and $\mathbb{P}^3$*

A number of results concerning orthogonal involutions are gathered together for future reference. It is possible to prove these results algebraically, without explicit reference to the absolute conic; however, a geometrical style of argument is adopted, since this clarifies the connections between these results and the general theory of projective geometry.

Theorem 4.8. *Let two distinct points in the plane at infinity be given such that neither point lies on the absolute conic, and such that the line joining the two points is not tangent to the absolute conic. Then there are exactly two orthogonal involutions of the plane at infinity that interchange the two points.*

Proof. Let $\boldsymbol{m}, \boldsymbol{n}$ be distinct points of Π_∞, not contained in Ω, such that $\langle \boldsymbol{m}, \boldsymbol{n} \rangle$ is not tangent to Ω, and let $\boldsymbol{i}, \boldsymbol{j}$ be the points at which $\langle \boldsymbol{m}, \boldsymbol{n} \rangle$ intersects Ω. It follows from theorem 4.2 that there is a unique involution σ of $\langle \boldsymbol{m}, \boldsymbol{n} \rangle$ such that $\sigma \boldsymbol{m} = \boldsymbol{n}$, $\sigma \boldsymbol{i} = \boldsymbol{j}$. Let $\boldsymbol{p}_1, \boldsymbol{p}_2$ be fixed points of σ, and let τ_1, τ_2 be orthogonal involutions of Π_∞ with vertices at $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ respectively. It follows from theorem 4.2 that the restriction of each τ_i to $\langle \boldsymbol{m}, \boldsymbol{n} \rangle$ is equal to σ, because $\tau_i \boldsymbol{i} = \boldsymbol{j}$, and because each τ_i shares a fixed point on $\langle \boldsymbol{m}, \boldsymbol{n} \rangle$ with σ. It follows that $\tau_1 \boldsymbol{m} = \boldsymbol{n}$, $\tau_2 \boldsymbol{m} = \boldsymbol{n}$.

The proof is completed by showing that τ_1 and τ_2 are the only orthogonal involutions of Π_∞ that interchange $\boldsymbol{m}$ and $\boldsymbol{n}$. Let τ be any orthogonal involution of Π_∞ such that $\tau \boldsymbol{m} = \boldsymbol{n}$. It follows from theorem 4.4 that the vertex $\boldsymbol{p}_\tau$ of τ lies on $\langle \boldsymbol{m}, \boldsymbol{n} \rangle$, and it follows from the Corollary to theorem 4.5 that $\tau \boldsymbol{i} = \boldsymbol{j}$. By theorem 4.2 the restriction of τ to $\langle \boldsymbol{m}, \boldsymbol{n} \rangle$ is equal to σ, thus either $\boldsymbol{p}_\tau = \boldsymbol{p}_1$ or $\boldsymbol{p}_\tau = \boldsymbol{p}_2$. In the first case $\tau = \tau_1$ and in the second case $\tau = \tau_2$. □

If a cartesian coordinate system is chosen then the fixed points $\boldsymbol{p}_i$ of σ in the proof of theorem 4.8 are the external and internal bisectors of the angles between $\boldsymbol{m}$ and $\boldsymbol{n}$.

Theorem 4.9. *Let two distinct points be given in the plane at infinity such that the points are not orthogonal and such that neither point is contained in the absolute conic. Then there is exactly one non-trivial orthogonal involution of the plane at infinity that fixes both points.*

Proof. Let $\boldsymbol{m}, \boldsymbol{n}$ be distinct non-orthogonal points of Π_∞ such that neither point is contained in Ω. The orthogonal involution τ of Π_∞ with a line $g_\tau = \langle \boldsymbol{m}, \boldsymbol{n} \rangle$ of fixed points necessarily fixes $\boldsymbol{m}$ and $\boldsymbol{n}$. It remains to show that no other non-trivial orthogonal involution of Π_∞ fixes $\boldsymbol{m}$ and $\boldsymbol{n}$.

Let σ be a non-trivial orthogonal involution of Π_∞ with vertex $\boldsymbol{p}_\sigma$ and a line of fixed points g_σ such that $\boldsymbol{m}, \boldsymbol{n}$ are fixed points of σ. It follows from the proof of theorem 4.5 that if $\boldsymbol{m} = \boldsymbol{p}_\sigma$ then $\boldsymbol{n}$ is on the polar line of $\boldsymbol{m}$ with respect to Ω. In this case $\boldsymbol{n}$ is orthogonal to $\boldsymbol{m}$, contrary to hypothesis. Thus $\boldsymbol{m} \neq \boldsymbol{p}_\sigma$. Similarly, $\boldsymbol{n} \neq \boldsymbol{p}_\sigma$. It follows that $\boldsymbol{m}, \boldsymbol{n}$ are contained in g_σ, thus $g_\sigma = g_\tau$, and hence $\sigma = \tau$. □

Theorem 4.10. *A non-trivial orthogonal skew involution of $\mathbb{P}^3$ is uniquely determined by the line of fixed points of the involution not contained in the plane at infinity.*

Proof. Let τ be a non-trivial orthogonal skew involution of $\mathbb{P}^3$, and let g_τ, h_τ be the two skew lines of fixed points of τ, chosen such that h_τ is not included in Π_∞. The involution τ induces a non-trivial orthogonal involution of Π_∞ with a line of fixed points g_τ, and vertex $h_\tau \cap \Pi_\infty$. It follows from the proof of theorem 4.5 that g_τ is the polar of $h_\tau \cap \Pi_\infty$ with respect to Ω. Thus h_τ determines g_τ. The result follows because the fixed points of τ completely determine τ. □

Theorem 4.10 is simply a restatement of the result that a rotation through 180° is uniquely determined by its axis.

The next result is a useful criterion for showing that certain involutions are orthogonal.

Theorem 4.11. *A non-trivial involution of the plane at infinity is orthogonal if it fixes a point not on the absolute conic and if it also interchanges the points of contact of the tangents drawn from the fixed point to the absolute conic.*

Proof. Let τ be a non-trivial involution of Π_∞ with a fixed point $\boldsymbol{n}$ not on Ω such that τ interchanges the two points of contact $\boldsymbol{i}_n, \boldsymbol{j}_n$ of the tangents drawn from $\boldsymbol{n}$ to Ω. Let g_τ be the line of fixed points of τ. The conic $\tau(\Omega)$ contains $g_\tau \cap \Omega$, and in addition, $\tau(\Omega)$ is tangent to $\langle \boldsymbol{n}, \boldsymbol{i}_n \rangle$ and $\langle \boldsymbol{n}, \boldsymbol{j}_n \rangle$ at $\boldsymbol{i}_n$ and $\boldsymbol{j}_n$ respectively. A general conic has five degrees of freedom. Each tangency imposes two linear constraints on $\tau(\Omega)$, thus there remains only a one-parameter family of possibilities for $\tau(\Omega)$. The additional condition $g_\tau \cap \tau(\Omega) = g_\tau \cap \Omega$ ensures that $\tau(\Omega) = \Omega$. Hence τ is an orthogonal involution. □

The importance of involutions in the study of ambiguity arises from the fact that an orthogonal collineation (rotation) is expressible as a product of two orthogonal involutions in an infinite number of ways. This result is proved by Semple & Kneebone (1952) as part of the general theory of conics. An alternative proof is given as follows.

Theorem 4.12. *An orthogonal collineation of the plane at infinity with exactly one fixed point not on the absolute conic is expressible as the product of two orthogonal involutions with vertices on the polar line of the fixed point with respect to the absolute conic. Any point on the polar line, but not on the absolute conic, can serve as the vertex of either one of the involutions. The vertex of the other involution is then uniquely determined.*

Proof. Let ω be an orthogonal collineation of Π_∞ with fixed point $\boldsymbol{n}$ not on Ω, and let τ be an orthogonal involution of Π_∞ with vertex $\boldsymbol{p}_\tau$ on the polar line, $\pi\boldsymbol{n}$, of $\boldsymbol{n}$ with respect to Ω. Then $\tau\boldsymbol{n} = \boldsymbol{n}$, thus $\omega\tau\boldsymbol{n} = \boldsymbol{n}$. Let $\boldsymbol{i}_n, \boldsymbol{j}_n$ be the points at which $\pi\boldsymbol{n}$ intersects Ω. It follows from the corollary to theorem 4.5 that $\tau\boldsymbol{i}_n = \boldsymbol{j}_n$, thus by theorem 3.2, $\omega\tau\boldsymbol{i}_n = \boldsymbol{j}_n$.

Define σ by $\sigma = \omega\tau$. It follows from theorem 3.2 that σ is an involution, thus $\omega = \sigma\tau$ is a product of involutions. The involution σ is uniquely determined by ω and τ.

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Let $\boldsymbol{p}_\sigma$ be the vertex of σ. It follows from the definition of σ that $\sigma \boldsymbol{i}_n = \boldsymbol{j}_n$. Thus $\pi \boldsymbol{n} = \langle \boldsymbol{i}_n, \boldsymbol{j}_n \rangle$ is invariant under σ. The restriction of σ to $\pi \boldsymbol{n}$ is not the identity. It follows that $\boldsymbol{p}_\sigma$ is on $\pi \boldsymbol{n}$.

A similar argument applied to ω^{-1} establishes that the vertex of σ can be chosen to be any point on $\pi \boldsymbol{n}$, in which case ω and σ uniquely determine τ. □

4.4. *Involutions of twisted cubics*

The properties of involutions of twisted cubics are summarized briefly. Full details can be found in Semple & Kneebone (1952). An involution of a twisted cubic, c, is defined to be an involution of the space $\mathbb{P}^3$ containing c that leaves c invariant. There are close connections between involutions of twisted cubics and involutions of $\mathbb{P}^1$, because each twisted cubic has a natural structure as a one-dimensional projective space.

Let c be a twisted cubic with parametrization $t \mapsto A(1, t, t^2, t^3)^{\mathrm{T}}$ where $(1, t)$ is a general point of $\mathbb{P}^1$ and A is a non-singular 4×4 matrix. Each involution τ of c induces a unique involution of $\mathbb{P}^1$ of the form

$$\begin{pmatrix} 1 \\ t \end{pmatrix} \mapsto \begin{pmatrix} a & d \\ c & b \end{pmatrix} \begin{pmatrix} 1 \\ t \end{pmatrix}, \tag{22}$$

where a, b, c, d depend on τ. Conversely, an involution of $\mathbb{P}^1$ of the form (22) yields a unique involution of c, and hence of the space $\mathbb{P}^3$ containing c. Thus if $\boldsymbol{a}, \boldsymbol{b}$ are any two distinct points of c, then there is a unique involution of c that fixes $\boldsymbol{a}$ and $\boldsymbol{b}$.

The situation for collineations is similar. Each collineation of c induces a unique collineation of $\mathbb{P}^1$ of the form (22). Conversely, each collineation of $\mathbb{P}^1$ gives rise to a unique collineation of $\mathbb{P}^3$ that leaves c invariant.

5. Rectangular quadrics

The rigidity constraint in the reconstruction problem ensures that ambiguous surfaces are a subclass of the quadrics known as rectangular quadrics. In this section it is shown that ambiguous surfaces are rectangular. Some of the properties of rectangular quadrics are obtained, with particular emphasis on the rigid involutions of rectangular quadrics.

Theorem 5.1. *The intersection of an ambiguous surface with the plane at infinity contains three points such that one of these points is not on the absolute conic, and such that the other two points are the points of contact of the tangents drawn from the first point to the absolute conic. The point not on the absolute conic is real. The other two points are complex conjugate.*

Proof. Let ψ be an ambiguous surface, let s_∞ be the conic $\psi \cap \Pi_\infty$, and let cartesian coordinates be chosen such that ψ is given by an equation of the form (7). It is shown in §2.1 that s_∞ consists of points $\boldsymbol{x}$ in Π_∞ satisfying the equation

$$(U\boldsymbol{x} \times \boldsymbol{x}) \cdot \boldsymbol{b} = 0, \tag{23}$$

where U is an orthogonal matrix. Let $\boldsymbol{n}$ be the axis of U, and let $\boldsymbol{i}_n, \boldsymbol{j}_n$ be the points of contact of the tangents drawn from $\boldsymbol{n}$ to the absolute conic Ω, as shown in figure 8. The points $\boldsymbol{n}$, $\boldsymbol{i}_n, \boldsymbol{j}_n$ are the three points referred to in the statement of the theorem. Each point is in s_∞ because, by theorem 3.2, $\boldsymbol{n}$, $\boldsymbol{i}_n, \boldsymbol{j}_n$ are eigenvectors of U, and thus satisfy the equations

$$U\boldsymbol{n} \times \boldsymbol{n} = U\boldsymbol{i}_n \times \boldsymbol{i}_n = U\boldsymbol{j}_n \times \boldsymbol{j}_n = 0. \tag{24}$$

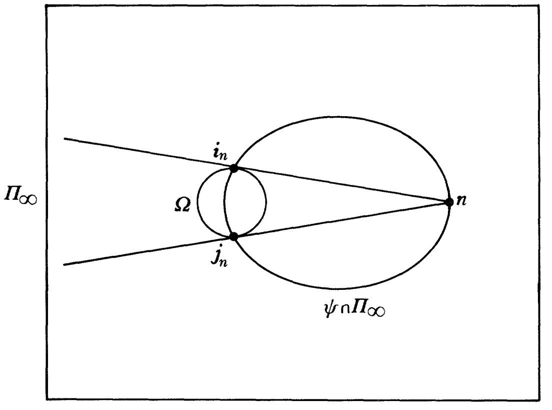

Figure 8. Rectangular quadrics.

The point $\boldsymbol{n}$ has real coordinates, and $\boldsymbol{i}_n, \boldsymbol{j}_n$ are complex conjugate because U is an orthogonal matrix with real entries. □

Theorem 5.1 leads to the following definitions.

Definition 5.1. A principal point of a quadric is a real point contained in the intersection of the quadric with the plane at infinity such that the quadric contains the two points of contact of the tangents drawn from the principal point to the absolute conic.

Definition 5.2. A real quadric is rectangular if and only if (i) it contains a principal point; and (ii) the quadric is not tangent to the absolute conic.

It is a consequence of definition 5.2 that a rectangular quadric intersects the absolute conic at four distinct points.

It is specified in definitions 5.1 and 5.2 that the principal point of a rectangular quadric are real, since this is the case of most relevance to the study of ambiguity. However, several of the properties of rectangular quadrics obtained below hold even if the principal points of a rectangular quadric are allowed to have complex coordinates.

It follows from theorem 5.1 and definition 5.2 that ambiguous surfaces are rectangular quadrics. Although the proof of theorem 5.1 involves choosing cartesian coordinates, the definition of a principal point, and hence also the definition of a rectangular quadric, is geometric, and does not depend on the choice of a particular coordinate system, provided the sets of real points and complex points are unchanged.

5.1. *Properties of rectangular quadrics*

Theorem 5.2. *A real rectangular quadric has two principal points.*

Proof. Let ψ be a real rectangular quadric with a principal point $\boldsymbol{n}$ and let $\boldsymbol{i}_n, \boldsymbol{j}_n$ be the points of contact of the tangents drawn from $\boldsymbol{n}$ to the absolute conic Ω, as illustrated in figure 9. It follows from definition 5.1 that $\boldsymbol{i}_n, \boldsymbol{j}_n$ are contained in the conic $s_\infty = \psi \cap \Pi_\infty$.

Let $\boldsymbol{i}_m, \boldsymbol{j}_m$ be the points of $s_\infty \cap \Omega$ distinct from $\boldsymbol{i}_n, \boldsymbol{j}_n$, and let $\boldsymbol{m}$ be the pole of $\langle \boldsymbol{i}_m, \boldsymbol{j}_m \rangle$ with respect to Ω. The point $\boldsymbol{m}$ is real because $\boldsymbol{i}_m, \boldsymbol{j}_m$ are complex conjugate. To prove the theorem it is sufficient to show that $\boldsymbol{m}$ is contained in s_∞.

Let τ be the orthogonal involution of Π_∞ with vertex $\boldsymbol{p}_\tau$ at the intersection of the

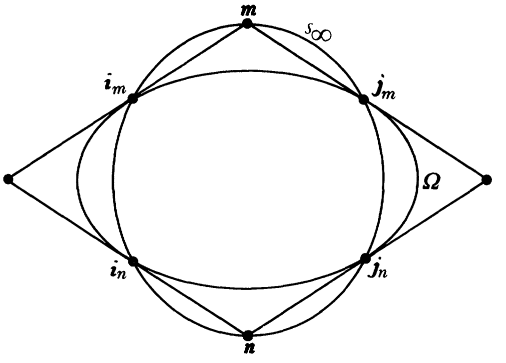

Figure 9. Illustration of theorem 5.2.

two lines $\langle \boldsymbol{i}_n, \boldsymbol{i}_m\rangle$ and $\langle \boldsymbol{j}_n, \boldsymbol{j}_m\rangle$. Then $\tau\boldsymbol{i}_n = \boldsymbol{i}_m$ and $\tau\boldsymbol{j}_n = \boldsymbol{j}_m$. Let g_τ be the line of fixed points of τ. Then $\tau(s_\infty) \cap s_\infty$ contains $\boldsymbol{i}_n, \boldsymbol{j}_n, \boldsymbol{i}_m, \boldsymbol{j}_m$ and the points of $g_\tau \cap s_\infty$. A conic has only five degrees of freedom, thus $\tau(s_\infty) = s_\infty$. It follows that $\boldsymbol{m} = \tau\boldsymbol{n}$ is a point of s_∞. Thus ψ has two principal points, as required. □

Theorem 5.3. *The tangent planes to a rectangular quadric at the two principal points intersect in a line orthogonal to both principal points of the quadric.*

Proof. Let ψ be a rectangular quadric with principal points $\boldsymbol{m}, \boldsymbol{n}$, let s_∞ be the conic $\psi \cap \Pi_\infty$, and let g be the line $\langle \boldsymbol{m}, \boldsymbol{n}\rangle$. Let the line formed by the intersection of the tangent planes to ψ at $\boldsymbol{m}$, $\boldsymbol{n}$ meet Π_∞ at $\boldsymbol{p}$. Then it is required to show that $\boldsymbol{p}$ is orthogonal to $\boldsymbol{m}$, $\boldsymbol{n}$, or equivalently, that $\boldsymbol{p}$ is the pole of g with respect to Ω.

Let τ be the rigid involution of Π_∞ with line of fixed points g. Then $\tau(s_\infty) \cap s_\infty$ contains the six points $\{\boldsymbol{m}, \boldsymbol{i}_m, \boldsymbol{j}_m, \boldsymbol{n}, \boldsymbol{i}_n, \boldsymbol{j}_n\}$, thus $\tau(s_\infty) = s_\infty$. It follows from the proof of theorem 4.7 that the vertex of τ is $\boldsymbol{p}$. The involution τ leaves Ω invariant, thus the vertex of τ is also the pole of $\langle \boldsymbol{m}, \boldsymbol{n}\rangle$ with respect to Ω. It follows that $\boldsymbol{p}$ is orthogonal to $\boldsymbol{m}$ and $\boldsymbol{n}$. □

5.2. *Characterizations of rectangular quadrics*

Several useful characterizations of rectangular quadrics are obtained under the assumption that cartesian coordinates have been chosen. Additional results are given by Horn (1990) and Negahdaripour (1990).

Theorem 5.4. *Let cartesian coordinates be chosen in* $\mathbb{P}^3$. *Then a quadric is rectangular if and only if it intersects the plane at infinity in a conic with an equation* $\boldsymbol{x}^{\mathrm{T}} M_{mn} \boldsymbol{x} = 0$, *where*

$$M_{mn} \equiv \tfrac{1}{2}(\boldsymbol{m} \otimes \boldsymbol{n} + \boldsymbol{n} \otimes \boldsymbol{m}) - \boldsymbol{m}\cdot\boldsymbol{n} I \tag{25}$$

and $\boldsymbol{m}$, $\boldsymbol{n}$ *are real points of* Π_∞.

Proof. Let ψ be a rectangular quadric, let s_∞ be the conic $\psi \cap \Pi_\infty$, and let $\boldsymbol{m}$, $\boldsymbol{n}$ be the principal points of ψ. It follows from theorem 5.1 and the equation $\boldsymbol{x}\cdot\boldsymbol{x} = 0$ for the absolute conic Ω that s_∞ contains distinct points $\boldsymbol{m}, \boldsymbol{i}_m, \boldsymbol{j}_m$ and $\boldsymbol{n}, \boldsymbol{i}_n, \boldsymbol{j}_n$, such that $\boldsymbol{m}$, $\boldsymbol{n}$ are real, and such that

$$\boldsymbol{m}\cdot\boldsymbol{i}_m = \boldsymbol{m}\cdot\boldsymbol{j}_m = \boldsymbol{i}_m\cdot\boldsymbol{i}_m = \boldsymbol{j}_m\cdot\boldsymbol{j}_m = 0, \tag{26}$$

$$\boldsymbol{n}\cdot\boldsymbol{i}_n = \boldsymbol{n}\cdot\boldsymbol{j}_n = \boldsymbol{i}_n\cdot\boldsymbol{i}_n = \boldsymbol{j}_n\cdot\boldsymbol{j}_n = 0. \tag{27}$$

Let s be the conic of Π_∞ defined by $\boldsymbol{x}^{\mathrm{T}} M_{mn} \boldsymbol{x} = 0$. It follows from (25), (26) and (27) that the six points $\boldsymbol{m}$, $\boldsymbol{i}_m, \boldsymbol{j}_m$, $\boldsymbol{n}$, $\boldsymbol{i}_n, \boldsymbol{j}_n$ belong to both s and s_∞, thus $s = s_\infty$.

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Conversely, let a quadric, ψ, intersect Π_∞ in a conic s_∞ of the form $\boldsymbol{x}^{\mathrm{T}} M_{mn}\boldsymbol{x} = 0$. It follows that the equation for s_∞ is $(\boldsymbol{x}\cdot\boldsymbol{m})(\boldsymbol{x}\cdot\boldsymbol{n})-(\boldsymbol{m}\cdot\boldsymbol{n})(\boldsymbol{x}\cdot\boldsymbol{x}) = 0$. The conic s_∞ contains six points $\boldsymbol{m}$, $\boldsymbol{i}_m, \boldsymbol{j}_m$, $\boldsymbol{n}$, $\boldsymbol{i}_n, \boldsymbol{j}_n$ satisfying (26) and (27). Hence $\boldsymbol{m}$, $\boldsymbol{n}$ are principal points of ψ. □

Theorem 5.5. *Let cartesian coordinates be chosen in $\mathbb{P}^3$. Then a quadric is rectangular if and only if it intersects the plane at infinity in a conic with an equation $\boldsymbol{x}^{\mathrm{T}}M\boldsymbol{x} = 0$, where M is a real symmetric matrix such that two of the eigenvalues of M have opposite signs, and sum to give the third eigenvalue of M.*

Proof. Let ψ be a non-singular quadric, and let s_∞ be the conic $\psi \cap \Pi_\infty$. It follows from theorem 5.4 that ψ is rectangular if and only if there exist real points $\boldsymbol{m}$, $\boldsymbol{n}$ in Π_∞ such that s_∞ has an equation $\boldsymbol{x}^{\mathrm{T}} M_{mn}\boldsymbol{x} = 0$. It suffices to show that a real symmetric matrix M has the form M_{mn} for real points $\boldsymbol{m}$, $\boldsymbol{n}$ in Π_∞ if and only if two of the eigenvalues of M have opposite signs, and sum to give the third eigenvalue of M.

It follows from (25) that the (unnormalized) eigenvectors of M_{mn} are

$$\|\boldsymbol{n}\|\,\boldsymbol{m}+\|\boldsymbol{m}\|\,\boldsymbol{n},\quad \|\boldsymbol{n}\|\,\boldsymbol{m}-\|\boldsymbol{m}\|\,\boldsymbol{n},\quad \boldsymbol{m}\times\boldsymbol{n},$$

where $\|\cdot\|$ is the euclidean norm of a vector. The corresponding eigenvalues are

$$\tfrac{1}{2}(\|\boldsymbol{m}\|\,\|\boldsymbol{n}\|-\boldsymbol{m}\cdot\boldsymbol{n}),\quad -\tfrac{1}{2}(\|\boldsymbol{m}\|\,\|\boldsymbol{n}\|+\boldsymbol{m}\cdot\boldsymbol{n}),\quad -\boldsymbol{m}\cdot\boldsymbol{n}. \tag{28}$$

The first eigenvalue in (28) is positive, the second eigenvalue is negative, and the third eigenvalue is the sum of the first two eigenvalues.

Conversely, let M be a real symmetric matrix with eigenvalues λ_1, λ_2, λ_3 such that $\lambda_1+\lambda_2 = \lambda_3$, and such that $\lambda_1 < 0 < \lambda_2$. Let $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$ be the corresponding eigenvectors, scaled such that $\boldsymbol{e}_1\cdot\boldsymbol{e}_1 = \boldsymbol{e}_2\cdot\boldsymbol{e}_2 = \boldsymbol{e}_3\cdot\boldsymbol{e}_3$. Define vectors $\boldsymbol{m}$, $\boldsymbol{n}$ by

$$\boldsymbol{m} = -\alpha\boldsymbol{e}_1+\beta\boldsymbol{e}_2,\quad \boldsymbol{n} = \alpha\boldsymbol{e}_1-\beta\boldsymbol{e}_2, \tag{29}$$

where $\alpha = \surd\lambda_2$, $\beta = \surd-\lambda_1$. It follows from (25) and (29) that

$$M_{mn} = -\lambda_2\,\boldsymbol{e}_1\otimes\boldsymbol{e}_1-\lambda_1\,\boldsymbol{e}_2\otimes\boldsymbol{e}_2+(\lambda_1+\lambda_2)\,I,$$

thus $M_{mn}\boldsymbol{e}_i = \lambda_i\boldsymbol{e}_i$. The matrices M_{mn} and M have the same eigenvectors and eigenvalues, thus $M_{mn} = M$. The proof is completed by noting that $\boldsymbol{m}$, $\boldsymbol{n}$, as defined in (29), are both real. □

Corollary. *If one of the eigenvalues of M is the sum of the other two eigenvalues of M, and if in addition ψ has a real generator then ψ is rectangular. The existence of a real generator ensures that the eigenvalues of M do not all have the same sign. It follows that two eigenvalues with different signs sum to give the third eigenvalue.*

5.3. *Rigid involutions of rectangular quadrics*

It is shown that a rectangular quadric is invariant under exactly three non-trivial rigid skew involutions, such that two of the involutions interchange the principal points of the rectangular quadric, and the third involution fixes the principal points. This result is applied in §7.1 to show that each ambiguous surface is invariant under a rigid involution that interchanges the two possible positions of the optical centre of the camera from which the second image is obtained.

The rigid skew involutions are defined as follows. Let $\boldsymbol{m}$, $\boldsymbol{n}$ be the principal points of a rectangular quadric ψ, and let l_ψ be the line formed by the intersection of the

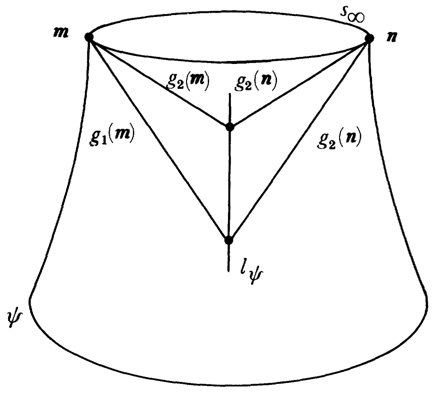

Figure 10. The geometry of rectangular quadrics.

tangent planes to ψ at $\boldsymbol{m}$ and $\boldsymbol{n}$. Define τ_ψ to be the non-trivial involution of $\mathbb{P}^3$ with two lines of fixed points l_ψ and $\langle \boldsymbol{m}, \boldsymbol{n} \rangle$. It follows from theorem 5.3 that $l_\psi \cap \Pi_\infty$ is orthogonal to every point of $\langle \boldsymbol{m}, \boldsymbol{n} \rangle$, thus τ_ψ is rigid.

Let $\boldsymbol{p}_1$, $\boldsymbol{p}_2$ be the fixed points of the unique rigid involution of the line $\langle \boldsymbol{m}, \boldsymbol{n} \rangle$ that interchanges $\boldsymbol{m}$ and $\boldsymbol{n}$, and let $\boldsymbol{d}$ be the mid-point of the line segment defined by the two intersections of l_ψ with ψ. Define τ_1 to be the rigid skew involution of $\mathbb{P}^3$ with a line of fixed points $\langle \boldsymbol{d}, \boldsymbol{p}_1 \rangle$, and define τ_2 to be the rigid skew involution of $\mathbb{P}^3$ with a line of fixed points $\langle \boldsymbol{d}, \boldsymbol{p}_2 \rangle$.

It follows from the definitions of τ_ψ, τ_1, τ_2, that $\tau_\psi \boldsymbol{m} = \boldsymbol{m}$, $\tau_\psi \boldsymbol{n} = \boldsymbol{n}$, and $\tau_i \boldsymbol{m} = \boldsymbol{n}$ for $i = 1, 2$. It is shown in the next three theorems that τ_ψ, τ_1, τ_2 are precisely the non-trivial rigid skew involutions of ψ.

Theorem 5.6. *A non-singular rectangular quadric is invariant under three non-trivial rigid skew involutions.*

Proof. It is shown that a non-singular rectangular quadric ψ is invariant under the three rigid skew involutions τ_ψ, τ_1, τ_2. Let $\boldsymbol{m}$, $\boldsymbol{n}$ be the principal points of ψ, and let l_ψ be the line formed by the intersection of the tangent planes to ψ at $\boldsymbol{m}$ and $\boldsymbol{n}$. Let $g_1(\boldsymbol{m})$, $g_2(\boldsymbol{m})$ be the generators of ψ through $\boldsymbol{m}$, as shown in figure 10. The tangent plane to ψ at $\boldsymbol{m}$ contains $g_1(\boldsymbol{m})$, thus $g_1(\boldsymbol{m})$ intersects l_ψ. The line $g_1(\boldsymbol{m})$ contains two fixed points of τ_ψ, namely $\boldsymbol{m}$ and $g_1(\boldsymbol{m}) \cap l_\psi$, thus $g_1(\boldsymbol{m})$ is invariant under τ_ψ. Hence $\tau_\psi(\psi)$ contains $g_1(\boldsymbol{m})$. Similarly, $\tau_\psi(\psi)$ contains $g_2(\boldsymbol{m})$, $g_1(\boldsymbol{n})$ and $g_2(\boldsymbol{n})$.

The generators $g_i(\boldsymbol{m})$, $g_i(\boldsymbol{n})$ meet l_ψ at the points of $l_\psi \cap \psi$. It follows from the definitions of τ_1 and τ_2 that the τ_i interchange the two points of $l_\psi \cap \psi$, thus the τ_i permute the four lines $g_i(\boldsymbol{m})$, $g_i(\boldsymbol{n})$ among themselves. It follows that the $\tau_i(\psi)$ contain $g_1(\boldsymbol{m})$, $g_2(\boldsymbol{m})$ and $g_1(\boldsymbol{n})$, $g_2(\boldsymbol{n})$.

Let $s_\infty = \psi \cap \Pi_\infty$. Each of the three intersections $\tau_\psi(s_\infty) \cap s_\infty$, $\tau_1(s_\infty) \cap s_\infty$, $\tau_2(s_\infty) \cap s_\infty$ contains the six points $\{\boldsymbol{m}, \boldsymbol{i}_m, \boldsymbol{j}_m, \boldsymbol{n}, \boldsymbol{i}_n, \boldsymbol{j}_n\}$. Two distinct conics intersect at four points only, thus the conics in each of the three intersections are not distinct. In other words, $\tau_\psi(s_\infty) = s_\infty$, $\tau_1(s_\infty) = s_\infty$, $\tau_2(s_\infty) = s_\infty$.

It has been shown that each of the three intersections $\tau_\psi(\psi) \cap \psi$, $\tau_1(\psi) \cap \psi$, $\tau_2(\psi) \cap \psi$ contains a (split) space curve of degree six, namely the union of s_∞ with the four generators $g_i(\boldsymbol{m})$, $g_i(\boldsymbol{n})$. The intersection of two distinct quadrics is a space curve of degree four rather than degree six, thus the quadrics in each of the pairs $\{\tau_\psi(\psi), \psi\}$, $\{\tau_1(\psi), \psi\}$, $\{\tau_2(\psi), \psi\}$ are not distinct. In other words $\tau_\psi(\psi) = \psi$, $\tau_1(\psi) = \psi$, $\tau_2(\psi) = \psi$. □

The generators $g_i(\boldsymbol{m})$, $g_i(\boldsymbol{n})$ in the proof of theorem 5.6 are referred to by Hofmann (1950) as the main generators (Haupterzeugenden) of ψ. Any two generators g, h of ψ such that $\tau_\psi(g) = h$ are known as adjoint generators (Erzeugenden adjungiert). The main generators of ψ are thus self-adjoint generators. The involution τ_ψ first appeared in the photogrammetric literature. Further information and references are given by Hofmann (1950).

Theorem 5.7. *A non-trivial skew involution of* $\mathbb{P}^3$ *leaving invariant a non-singular quadric also leaves invariant the two families of generators on the quadric.*

Proof. Let ψ be a non-singular quadric invariant under a non-trivial skew involution τ, and let $\mathscr{F}_1$, $\mathscr{F}_2$ be the two families of generators of ψ. Two generators, g and h, of ψ intersect if and only if the generators $\tau(g)$ and $\tau(h)$ intersect. It follows that g, h are in the same family of generators if and only if $\tau(g)$, $\tau(h)$ are in the same family of generators, thus either τ interchanges $\mathscr{F}_1$ and $\mathscr{F}_2$, or τ leaves $\mathscr{F}_1$ and $\mathscr{F}_2$ invariant.

Let ψ contain a line g_τ of fixed points of τ, and suppose, after relabelling if necessary, that g_τ is in $\mathscr{F}_1$. Let a generator h of $\mathscr{F}_2$ intersect g_τ at $\boldsymbol{p}$. Then $\tau(h)$ also intersects g_τ at $\boldsymbol{p}$, thus $h = \tau(h)$. It follows that $\mathscr{F}_2$ is invariant under τ, and thus $\mathscr{F}_1$ is also invariant under τ.

Next, suppose that ψ does not contain a line of fixed points of τ. Choose five generators g_i of ψ from $\mathscr{F}_1$. If τ interchanges $\mathscr{F}_1$ and $\mathscr{F}_2$ then the points $\tau(g_i) \cap g_i$ are fixed points of τ contained in ψ. The fixed points of τ form two lines in $\mathbb{P}^3$ thus at least three of the $\tau(g_i) \cap g_i$ are on the same line of fixed points of τ. It follows that ψ contains a line of fixed points of τ, contrary to hypothesis. The involution τ thus does not interchange $\mathscr{F}_1$ and $\mathscr{F}_2$. Hence τ leaves $\mathscr{F}_1$ and $\mathscr{F}_2$ invariant. □

Theorem 5.8. *A non-singular rectangular quadric is invariant under exactly three non-trivial rigid skew involutions.*

Proof. It has been shown in theorem 5.6 that a non-singular rectangular quadric, ψ, is invariant under three non-trivial rigid skew involutions, τ_ψ, τ_1 and τ_2. It is thus sufficient to show that if ψ is invariant under a non-trivial rigid skew involution τ, then τ is equal to one of τ_ψ, τ_1, τ_2.

Let $\boldsymbol{m}$, $\boldsymbol{n}$ be the principal directions of ψ. The absolute conic Ω and the conic $s_\infty = \psi \cap \Pi_\infty$ are invariant under τ, thus either $\boldsymbol{m}$, $\boldsymbol{n}$ are fixed points of τ or $\boldsymbol{m}$, $\boldsymbol{n}$ are interchanged by τ. It follows from theorems 4.8 and 4.9 that the restriction of τ to Π_∞ is equal to the restriction of one of τ_ψ, τ_1, τ_2 to Π_∞. Thus, by forming the appropriate product $\tau\tau_\psi$, $\tau\tau_1$, $\tau\tau_2$, a euclidean transformation ω is obtained which fixes every point of Π_∞ and which leaves ψ invariant.

It follows from theorem 5.7 that the two skew involutions comprising ω each leave the two families $\mathscr{F}_1$ and $\mathscr{F}_2$ of generators of ψ invariant, thus ω also leaves $\mathscr{F}_1$ and $\mathscr{F}_2$ invariant. Each point of s_∞ lies on a single generator of $\mathscr{F}_1$, and the points of s_∞ are fixed points of ω, thus each individual generator of $\mathscr{F}_1$ is invariant under ω. Similarly, each generator of $\mathscr{F}_2$ is invariant under ω. Each point $\boldsymbol{p}$ of ψ is at the intersection of two generators $g_1(\boldsymbol{p})$, $g_2(\boldsymbol{p})$ of ψ belonging to $\mathscr{F}_1$ and $\mathscr{F}_2$ respectively, thus all the points of ψ are fixed by ω.

Let $\boldsymbol{a}$ be any point of $\mathbb{P}^3$, and let k be a line through $\boldsymbol{a}$. Then k contains three fixed points of ω, namely, $k \cap \Pi_\infty$ and the two points of $k \cap \psi$. It follows that k is invariant under ω. The restriction of ω to k is the identity, thus $\omega\boldsymbol{a} = \boldsymbol{a}$. It follows that ω is the identity. Thus τ is equal to one of τ_ψ, τ_1, τ_2, as required. □

6. Horopter curves

Up to this point the discussion of ambiguity has centred on the ambiguous surfaces, in line with much of the current computer vision work on reconstruction. There is, however, a second approach to ambiguity based on the properties of certain space curves of degree three known as horopter curves. Horopter curves were first discovered by Helmholtz (1925). They appear in the earlier projective geometric treatment of ambiguity (Wunderlich 1942), but with the exception of Buchanan's work (1987), they have not so far been used in computer vision. The first theorem of this section shows that the twisted cubic contained in the intersection of complementary ambiguous surfaces has a key property characterizing horopter curves.

Theorem 6.1. *The twisted cubic contained in the intersection of complementary ambiguous surfaces contains three points such that one of these points is not on the absolute conic and such that the other two points are the points of contact of the tangents drawn from the first point to the absolute conic. The first point is real and the second two points are complex conjugate.*

Proof. The result follows on applying theorem 5.1 to a pair ψ, ϕ of complementary ambiguous surfaces. The points $\boldsymbol{n}$, $\boldsymbol{i}_n$, $\boldsymbol{j}_n$, as given in theorem 5.1, are contained in $\psi \cap \phi$. They are not on the common generator of ψ, ϕ, thus $\boldsymbol{n}$, $\boldsymbol{i}_n$, $\boldsymbol{j}_n$ are contained in the twisted cubic common to ψ and ϕ. □

The twisted cubics contained in the intersection of complementary ambiguous surfaces are examples of horopter curves.

6.1. *Star generation of horopter curves*

Horopter curves arise in theorem 6.1 as part of the intersection of complementary ambiguous surfaces. It is, however, convenient to base the definition of horopter curves on the star generation of twisted cubics (Semple & Kneebone 1952), rather than on theorem 6.1.

Definition 6.1. Let two stars of lines with distinct real centres be given in $\mathbb{P}^3$, together with a real orthogonal collineation between the two stars of lines. Then the locus of the intersections of corresponding pairs of lines is defined to be a horopter curve.

It follows from definition 6.1 and the construction of ambiguous surfaces described in §2.3 that the twisted cubic contained in the intersection of complementary ambiguous surfaces is a horopter curve.

The equations defining a horopter curve are readily obtained from definition 6.1. Let $\boldsymbol{a}$, $\boldsymbol{b}$ be the two centres of the stars of lines, and let ω be the orthogonal collineation from the lines through $\boldsymbol{a}$ to the lines through $\boldsymbol{b}$. It follows from definition 6.1 that a point $\boldsymbol{x}$ is on the horopter curve if and only if the vectors $\omega(\boldsymbol{x}-\boldsymbol{a})$ and $\boldsymbol{x}-\boldsymbol{b}$ are parallel. In cartesian coordinates, $\boldsymbol{x}$ is on the horopter curve if and only if

$$\omega(\boldsymbol{x}-\boldsymbol{a}) \times (\boldsymbol{x}-\boldsymbol{b}) = 0. \tag{30}$$

The horopter curve c given by (30) intersects Π_∞ at the points $\boldsymbol{x}$ which are solutions of $\omega\boldsymbol{x} \times \boldsymbol{x} = 0$. The points of $c \cap \Pi_\infty$ are precisely the eigenvectors $\boldsymbol{n}$, $\boldsymbol{i}_n$, $\boldsymbol{j}_n$ of ω, as required by theorem 6.1. The following theorem shows that the converse to theorem 6.1 is also true.

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Theorem 6.2. *A non-singular real twisted cubic is a horopter curve if and only if it intersects the plane at infinity at three points such that one of these points is real, and such that the other two points are the points of contact of the two tangents drawn from the first point to the absolute conic.*

Proof. It is shown in the discussion following (30) that a non-singular horopter curve intersects Π_∞ at three points $\boldsymbol{n}$, $\boldsymbol{i}_n$, $\boldsymbol{j}_n$ satisfying the requirements of this theorem. Conversely, let c be a non-singular real twisted cubic intersecting Π_∞ at $\boldsymbol{n}$, $\boldsymbol{i}_n$, $\boldsymbol{j}_n$. Then $\boldsymbol{n}$ is real and $\boldsymbol{i}_n$, $\boldsymbol{j}_n$ are complex conjugate. It follows from the general properties of twisted cubics that there exists a unique involution τ of c (and hence of $\mathbb{P}^3$) such that $\tau\boldsymbol{n} = \boldsymbol{n}$ and $\tau\boldsymbol{i}_n = \boldsymbol{j}_n$. The plane $\Pi_\infty = \langle \boldsymbol{n}, \boldsymbol{i}_n, \boldsymbol{j}_n \rangle$ is invariant under τ. It follows from theorem 4.11 that τ is rigid. Let $\boldsymbol{a}$ be a real point of c such that $\tau\boldsymbol{a} \neq \boldsymbol{a}$, and let $\boldsymbol{b} = \tau\boldsymbol{a}$. It follows from the star generation of twisted cubics that there exists a collineation ω of Π_∞ such that a point $\boldsymbol{x}$ is on c if and only if there is a point $\boldsymbol{r}$ of Π_∞, depending on $\boldsymbol{x}$, such that

$$\boldsymbol{x} = \langle \boldsymbol{a}, \boldsymbol{r} \rangle \cap \langle \boldsymbol{b}, \omega\boldsymbol{r} \rangle. \tag{31}$$

As $\boldsymbol{x}$ varies, the points $\boldsymbol{r}$ of (31) vary on the conic s formed by projecting c from $\boldsymbol{a}$ to Π_∞.

The collineation ω is real because c is real. To see this, let $\bar{\boldsymbol{x}}$ be the complex conjugate of $\boldsymbol{x}$. Complex conjugation of (31) yields

$$\bar{\boldsymbol{x}} = \langle \boldsymbol{a}, \bar{\boldsymbol{r}} \rangle \cap \langle \boldsymbol{b}, \bar{\omega}\bar{\boldsymbol{r}} \rangle$$

The point $\bar{\boldsymbol{x}}$ is on c, thus $\omega\bar{\boldsymbol{r}} = \bar{\omega}\bar{\boldsymbol{r}}$. It follows that $\omega = \bar{\omega}$ as required.

To prove the theorem it suffices to show that ω is an orthogonal collineation. The application of τ to (31) yields

$$\tau\boldsymbol{x} = \langle \boldsymbol{b}, \tau\boldsymbol{r} \rangle \cap \langle \boldsymbol{a}, \tau\omega\boldsymbol{r} \rangle. \tag{32}$$

The point $\tau\boldsymbol{x}$ is on c, because c is invariant under τ. A comparison of (31) and (32) yields $\omega\tau\omega\boldsymbol{r} = \tau\boldsymbol{r}$ for all points $\boldsymbol{r}$ on s. It follows that $(\tau\omega)^2$ is the identity on s, thus $(\tau\omega)^2$ is the identity on the whole of Π_∞. Hence $\tau\omega$ is an involution of Π_∞. It follows from (31) that $\boldsymbol{n}$, $\boldsymbol{i}_n$, $\boldsymbol{j}_n$ are fixed points of ω, thus $\tau\omega\boldsymbol{n} = \boldsymbol{n}$ and $\tau\omega\boldsymbol{i}_n = \boldsymbol{j}_n$. It follows from theorem 4.11 that $\tau\omega$ is orthogonal. Hence ω is an orthogonal collineation. □

Corollary. *A horopter curve is invariant under a non-trivial rigid involution.*

An explicit construction of the non-trivial rigid involution appearing in the proof of theorem 6.2 is given in the next section.

6.2. *Rigid involutions of horopter curves*

Rigid involutions of horopter curves and ambiguous surfaces are the basis of the results on ambiguity obtained in §7. It has been shown in the corollary to theorem 6.2 that a horopter curve is invariant under a non-trivial rigid involution. This rigid involution is constructed explicitly in the following theorem.

Theorem 6.3. *A horopter curve is invariant under a non-trivial rigid skew involution.*

Proof. Let c be a horopter curve generated by a real orthogonal collineation ω from the star of lines with centre $\boldsymbol{a}$ to the star of lines with centre $\boldsymbol{b}$, where $\boldsymbol{a}$, $\boldsymbol{b}$ are distinct real points of $\mathbb{P}^3$. Let ω be described by its action on Π_∞. Let $\boldsymbol{n}$ be the axis of ω, let $\boldsymbol{p}$ be the intersection of $\langle \boldsymbol{a}, \boldsymbol{b} \rangle$ with Π_∞, and let $\boldsymbol{d}$ be the midpoint of the line segment $[\boldsymbol{a}, \boldsymbol{b}]$.

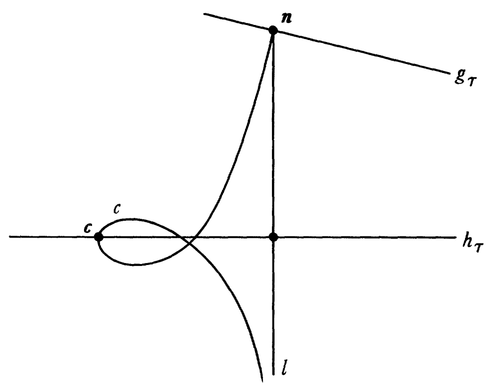

Figure 11. The rigid involution of a horopter curve.

The points of c are precisely those points $\boldsymbol{x}$ satisfying (31). It follows from theorem 4.12 that $\omega = \tau_1 \tau_2$ where τ_1 and τ_2 are rigid involutions of Π_∞ with vertices $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$ both orthogonal to $\boldsymbol{n}$. Set $\boldsymbol{n}_1 = \boldsymbol{n} \times \boldsymbol{p}$, and let τ be the rigid skew involution of $\mathbb{P}^3$ with a line of fixed points passing through $\boldsymbol{d}$ with direction $\boldsymbol{n}_1$. It follows from the definition of τ that $\tau\boldsymbol{a} = \boldsymbol{b}$, and that τ_1 is the restriction of τ to Π_∞. The application of τ to (31) yields

$$\begin{aligned} \tau\boldsymbol{x} &= \langle \boldsymbol{b}, \tau_1 \boldsymbol{r} \rangle \cap \langle \boldsymbol{a}, \tau_1 \omega \boldsymbol{r} \rangle \\ &= \langle \boldsymbol{b}, \tau_1 \boldsymbol{r} \rangle \cap \langle \boldsymbol{a}, \tau_2 \boldsymbol{r} \rangle \\ &= \langle \boldsymbol{a}, \boldsymbol{s} \rangle \cap \langle \boldsymbol{b}, \omega \boldsymbol{s} \rangle, \end{aligned}$$

where $\boldsymbol{s} = \tau_2 \boldsymbol{r}$. It follows that $\tau\boldsymbol{x}$ is a point of c. □

The rigid involution τ constructed in the proof of theorem 6.3 is illustrated in figure 11. The lines g_τ and h_τ are the two skew lines of fixed points of τ, labelled such that g_τ is contained in Π_∞. There are exactly two fixed points of τ on c. One of these points is the real point $\boldsymbol{n}$ of $c \cap \Pi_\infty$. The other point is known as the centre $\boldsymbol{c}$ of c. The point $\boldsymbol{n}$ lies on g_τ and $\boldsymbol{c}$ lies on h_τ. The tangent to c at $\boldsymbol{n}$ is known as the real asymptote of c. The tangents to c at $\boldsymbol{n}$ and $\boldsymbol{c}$ are both invariant under τ, thus each tangent is a common transversal of g_τ and h_τ.

Theorem 6.4. *A horopter curve is invariant under exactly one non-trivial rigid involution.*

Proof. It suffices to show that the rigid involution τ of the horopter curve c constructed in theorem 6.2 is unique. Let σ be any non-trivial rigid involution of c, and let $\boldsymbol{n}$, $\boldsymbol{i}_n, \boldsymbol{j}_n$ be the three points of $c \cap \Pi_\infty$. Both c and Π_∞ are invariant under σ, thus the set $\{\boldsymbol{n}, \boldsymbol{i}_n, \boldsymbol{j}_n\}$ is permuted by σ. In addition, $\sigma\boldsymbol{n} = \boldsymbol{n}$ because $\boldsymbol{n}$ is the only point of $c \cap \Pi_\infty$ not contained in Ω.

It follows from §4.4 that σ is uniquely determined by its action on $\{\boldsymbol{n}, \boldsymbol{i}_n, \boldsymbol{j}_n\}$. If $\sigma\boldsymbol{i}_n = \boldsymbol{i}_n$, $\sigma\boldsymbol{j}_n = \boldsymbol{j}_n$ then σ is the identity, contrary to hypothesis. The only remaining possibility is $\sigma\boldsymbol{i}_n = \boldsymbol{j}_n$. In this case σ and τ have the same effect on $\{\boldsymbol{n}, \boldsymbol{i}_n, \boldsymbol{j}_n\}$, thus $\sigma = \tau$. □

It is apparent from the proof of theorem 6.2 that there is a close connection between the star generation of a horopter curve by an orthogonal collineation and the unique non-trivial rigid involution of the horopter curve. The next theorem develops this connection further.

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Theorem 6.5. *Two distinct points of a horopter curve are centres of stars of lines related by an orthogonal collineation such that corresponding lines intersect on the horopter curve if and only if the unique non-trivial rigid involution of the horopter curve interchanges the two points.*

Proof. Let c be a horopter curve invariant under a non-trivial rigid involution τ, and let $\boldsymbol{a}$, $\boldsymbol{b}$ be distinct points of c. It follows from theorems 6.2 and 6.4 that if $\boldsymbol{a}$, $\boldsymbol{b}$ are centres of two stars of lines related by an orthogonal collineation that generates c then $\tau\boldsymbol{a} = \boldsymbol{b}$. To prove the converse, suppose that $\tau\boldsymbol{a} = \boldsymbol{b}$. Let ω be the collineation of Π_∞ such that if $\boldsymbol{x}$ is any point of c then there exists a point $\boldsymbol{r}$ of Π_∞, depending on $\boldsymbol{x}$, such that (31) holds. It is sufficient to prove that ω is orthogonal. Let τ_1 be the restriction of τ to Π_∞. Then the application of τ to (31) yields

$$\tau\boldsymbol{x} = \langle \boldsymbol{b}, \tau_1 \boldsymbol{r}\rangle \cap \langle \boldsymbol{a}, \tau_1 \omega\boldsymbol{r}\rangle. \tag{33}$$

By hypothesis, $\tau\boldsymbol{x}$ is a point of c. A comparison of equations (31) and (33) yields $\omega\tau_1\omega\boldsymbol{r} = \tau_1\boldsymbol{r}$ for all points $\boldsymbol{r}$ on the projection s of c from $\boldsymbol{a}$ to Π_∞. The curve s is a non-singular conic, thus $(\tau_1\omega)^2$ is the identity, and hence $\tau_1\omega$ is an involution. Let $c \cap \Pi_\infty = \{\boldsymbol{n}, \boldsymbol{i}_n, \boldsymbol{j}_n\}$ where $\boldsymbol{i}_n$, $\boldsymbol{j}_n$ are on the absolute conic Ω. The involution $\tau_1\omega$ fixes $\boldsymbol{n}$ and interchanges $\boldsymbol{i}_n$ and $\boldsymbol{j}_n$, thus by theorem 4.11, $\tau_1\omega$ is orthogonal. Hence ω is an orthogonal collineation. □

An alternative characterization of the centre of a horopter curve is now obtained.

Theorem 6.6. *A horopter curve contains a unique real point furthest from the real asymptotic line. This point is fixed by the unique non-trivial rigid involution of the horopter curve.*

Proof. Let c be a horopter curve invariant under a non-trivial rigid involution τ, and let $\boldsymbol{n}$, $\boldsymbol{i}_n$, $\boldsymbol{j}_n$ be the three intersections of c with Π_∞, such that $\boldsymbol{n}$ is not on the absolute conic Ω. Let l be the tangent to c at $\boldsymbol{n}$ (l is the real asymptotic line of c), and let ϕ_d be the circular cylinder of radius d with axis l. The surface ϕ_d is a quadric containing $\boldsymbol{n}$, $\boldsymbol{i}_n$, $\boldsymbol{j}_n$, with a singularity at $\boldsymbol{n}$. The line l is invariant under τ, thus ϕ_d is invariant under τ. It follows that $\phi_d \cap c$ is invariant under τ.

It is shown that for a general value of d, $\phi_d \cap c$ contains five distinct points. Let r be the value of d for which ϕ_d contains $\boldsymbol{a}$. Then ϕ_r also contains $\boldsymbol{b}$ because ϕ_r is invariant under τ, thus $c \cap \phi_r \supset \{\boldsymbol{n}, \boldsymbol{i}_n, \boldsymbol{j}_n, \boldsymbol{a}, \boldsymbol{b}\}$. It follows that $\phi_d \cap c$ contains in general at least five distinct points. The space curve c is of degree three and ϕ_d is of degree two thus $\phi_d \cap c$ contains $6 = 3 \times 2$ points, counted with the correct multiplicities (Walker 1962). The point $\boldsymbol{n}$ is counted at least twice in $\phi_d \cap c$ because ϕ_d has a singularity at $\boldsymbol{n}$. Thus $\phi_d \cap c$ contains in general at most five distinct points. In consequence, $\phi_d \cap c$ contains in general exactly five distinct points.

Let $\phi_d \cap c = \{\boldsymbol{n}, \boldsymbol{i}_n, \boldsymbol{j}_n, \boldsymbol{p}_d, \boldsymbol{q}_d\}$. Then $\tau\boldsymbol{p}_d = \boldsymbol{q}_d$ because τ has exactly two fixed points on c and in general neither of these fixed points is equal to $\boldsymbol{p}_d$ or $\boldsymbol{q}_d$. If d is sufficiently large then $\boldsymbol{p}_d$, $\boldsymbol{q}_d$ are complex conjugate. There also exist values of d for which $\boldsymbol{p}_d$ and $\boldsymbol{q}_d$ are both real, for example $d = r$. Let d' be the largest value of d for which $\boldsymbol{p}_d$, $\boldsymbol{q}_d$ are real. Then c is tangent to $\phi_{d'}$ at $\boldsymbol{p}_{d'}$, thus $\boldsymbol{p}_{d'}$ and $\boldsymbol{q}_{d'}$ coincide at a point $\boldsymbol{c}$, and $\tau\boldsymbol{c} = \boldsymbol{c}$. It follows from the definition of $\phi_{d'}$ that $\boldsymbol{c}$ is the unique point of c furthest from l. □

Theorem 6.6 is the basis of the following definition of the centre of a horopter curve.

Definition 6.2. The centre of a horopter curve is the unique point of the curve furthest from the real asymptote.

The centre of a horopter curve should not be confused with the optical centre of a camera or with the centre of a star of lines.

The next theorem is useful for the classification of horopter curves.

Theorem 6.7. *Let two horopter curves be given such that they are invariant under the same non-trivial rigid involution, they have the same centre, and they have the same intersections with the plane at infinity. If either* (i) *the two curves have at least one further point in common*; *or* (ii) *the two curves have the same real asymptotic line and the same tangent line at the common centre, then the two curves are identical.*

Proof. Let c_1, c_2 be two horopter curves invariant under the same non-trivial rigid involution τ, such that $c_1 \cap \Pi_\infty = c_2 \cap \Pi_\infty = \{\boldsymbol{n}, \boldsymbol{i}_n, \boldsymbol{j}_n\}$, and such that c_1, c_2 have the same centre $\boldsymbol{c}$. To prove (i), let $\boldsymbol{a}$ be a point common to c_1 and c_2, but distinct from $\boldsymbol{n}$, $\boldsymbol{i}_n$, $\boldsymbol{j}_n$ and $\boldsymbol{c}$. Let ω_1, ω_2 be collineations of Π_∞ that define c_1 and c_2 respectively as the loci of the intersections of corresponding lines from stars of lines centred at $\boldsymbol{a}$ and $\tau\boldsymbol{a}$. The horopter curves c_1 and c_2 intersect at $\boldsymbol{n}$, $\boldsymbol{i}_n$, $\boldsymbol{j}_n$ thus

$$\omega_1 \boldsymbol{n} = \omega_2 \boldsymbol{n} = \boldsymbol{n}, \quad \omega_1 \boldsymbol{i}_n = \omega_2 \boldsymbol{i}_n = \boldsymbol{i}_n, \quad \omega_1 \boldsymbol{j}_n = \omega_2 \boldsymbol{j}_n = \boldsymbol{j}_n. \tag{34}$$

Let $\boldsymbol{r} = \langle \boldsymbol{a}, \boldsymbol{c} \rangle \cap \Pi_\infty$, and let $\boldsymbol{s} = \langle \tau\boldsymbol{a}, \boldsymbol{c} \rangle \cap \Pi_\infty$. The centre $\boldsymbol{c}$ is on both c_1 and c_2 thus

$$\omega_1 \boldsymbol{r} = \omega_2 \boldsymbol{r} = \boldsymbol{s}. \tag{35}$$

It follows from (34) and (35) that ω_1 and ω_2 take the same values at four points of Π_∞, no three of which are collinear, thus $\omega_1 = \omega_2$, and hence $c_1 = c_2$.

To prove (ii), let c_1 and c_2 have the same real asymptotic line l and the same tangent line k at the common centre $\boldsymbol{c}$. Let ω_1, ω_2 be collineations from the star of lines centred at $\boldsymbol{c}$ to the star of lines centred at $\boldsymbol{n}$ that define c_1 and c_2 respectively as loci of the intersections of corresponding lines. Then

$$\omega_1\langle \boldsymbol{c}, \boldsymbol{i}_n \rangle = \omega_2\langle \boldsymbol{c}, \boldsymbol{i}_n \rangle = \langle \boldsymbol{n}, \boldsymbol{i}_n \rangle,$$
$$\omega_1\langle \boldsymbol{c}, \boldsymbol{j}_n \rangle = \omega_2\langle \boldsymbol{c}, \boldsymbol{j}_n \rangle = \langle \boldsymbol{n}, \boldsymbol{j}_n \rangle.$$

It follows from that star generation of twisted cubics that

$$\omega_1 k = \omega_2 k = \langle \boldsymbol{c}, \boldsymbol{n} \rangle, \quad \omega_2\langle \boldsymbol{c}, \boldsymbol{n} \rangle = \omega_2\langle \boldsymbol{c}, \boldsymbol{n} \rangle = l.$$

The collineations ω_1 and ω_2 agree at four points in their common domain of definition, thus $\omega_1 = \omega_2$, and hence $c_1 = c_2$. □

6.3. *Horopter curves on rectangular quadrics*

Each rectangular quadric contains numerous horopter curves. Among these, the horopter curves of greatest interest in reconstruction are those invariant under the unique non-trivial rigid involution of the rectangular quadric that fixes the principal points of the rectangular quadric. These horopter curves are now described.

Let $\boldsymbol{m}$, $\boldsymbol{n}$ be the principal points of a non-singular rectangular quadric ψ, and let τ_ψ be the unique non-trivial rigid skew involution of ψ that fixes $\boldsymbol{m}$ and $\boldsymbol{n}$. It follows from theorem 6.2 that any horopter curve in ψ contains either $\boldsymbol{m}$ or $\boldsymbol{n}$. Let $\mathscr{F}_1$, $\mathscr{F}_2$ be the two families of generators of ψ. Define $\mathscr{H}_1(\boldsymbol{n})$ to be the set of horopter curves contained in ψ and invariant under τ_ψ, that contain $\boldsymbol{n}$, and that meet each generator of $\mathscr{F}_1$ twice. The sets $\mathscr{H}_2(\boldsymbol{n})$, $\mathscr{H}_i(\boldsymbol{m})$ are defined *mutatis mutandis*. The four sets $\mathscr{H}_i(\boldsymbol{m})$, $\mathscr{H}_i(\boldsymbol{n})$ are disjoint, because a general horopter curve of ψ contains just one of the points $\boldsymbol{m}$, $\boldsymbol{n}$, and because each horopter curve in ψ is either a $(1,2)$ curve or a $(2,1)$ curve.

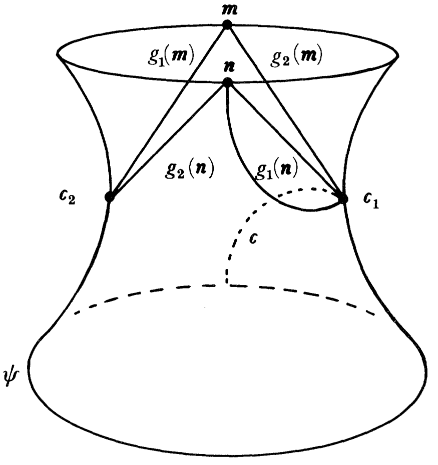

Figure 12. Horopter curves on a rectangular quadric.

Theorem 6.8. *The horopter curves contained in a non-singular rectangular quadric and invariant under the unique rigid skew involution of the quadric that fixes the two principal points of the quadric form four disjoint one-parameter families.*

Proof. The notation introduced immediately before the statement of this theorem is used. Let l_ψ be the line formed by the intersection of the tangent planes to the quadric ψ at $\boldsymbol{m}$ and $\boldsymbol{n}$, and let $\boldsymbol{c}_1$, $\boldsymbol{c}_2$ be the points at which l_ψ intersects ψ. The generators $g_1(\boldsymbol{n})$, $g_2(\boldsymbol{n})$ meet l_ψ. After relabelling $\boldsymbol{c}_1$ and $\boldsymbol{c}_2$ if necessary, it can be assumed that

$$g_1(\boldsymbol{n}) = \langle \boldsymbol{n}, \boldsymbol{c}_1 \rangle, \quad g_2(\boldsymbol{n}) = \langle \boldsymbol{n}, \boldsymbol{c}_2 \rangle$$

as illustrated in figure 12. It follows from the definition of τ_ψ, given in §5.3, that $\boldsymbol{c}_1$ and $\boldsymbol{c}_2$ are the only fixed points of τ_ψ contained in $\psi \backslash \Pi_\infty$, thus any horopter curve contained in ψ and invariant under τ_ψ has a centre either at $\boldsymbol{c}_1$ or at $\boldsymbol{c}_2$.

It is shown that all horopter curves c in $\mathscr{H}_1(\boldsymbol{n})$ have their centres at $\boldsymbol{c}_1$. It follows from the definition of $\mathscr{H}_1(\boldsymbol{n})$ that c meets $g_1(\boldsymbol{n})$ twice. Let $\boldsymbol{q}$ be the point of $c \cap g_1(\boldsymbol{n})$ distinct from $\boldsymbol{n}$. Then $\{\boldsymbol{n}, \boldsymbol{q}\}$ is invariant under τ_ψ. The point $\boldsymbol{n}$ is fixed by τ_ψ, thus $\boldsymbol{q}$ is fixed by τ_ψ. It follows that $\boldsymbol{q} = \boldsymbol{c}_1$, because $\boldsymbol{n}$, $\boldsymbol{c}_1$ are the only fixed points of τ_ψ in $g_1(\boldsymbol{n})$. Thus $\boldsymbol{c}_1$ is the centre of c.

Let $\boldsymbol{p}$ be an arbitrary point of ψ, and let c meet $g_2(\boldsymbol{p})$ at $\boldsymbol{a}$. It follows from theorem 6.7 that c is the only horopter curve in $\mathscr{H}_1(\boldsymbol{n})$ that contains $\boldsymbol{a}$. It remains to show that each point $\boldsymbol{a}$ of $g_2(\boldsymbol{p})$ lies on a horopter curve in $\mathscr{H}_1(\boldsymbol{n})$. Let ϕ be the unique cone with vertex $\boldsymbol{n}$ containing the points $S = \{\boldsymbol{n}, \boldsymbol{i}_n, \boldsymbol{j}_n, \boldsymbol{c}_1, \boldsymbol{a}, \tau_\psi \boldsymbol{a}\}$. The set S is invariant under τ_ψ and $\boldsymbol{n}$ is fixed by τ_ψ, thus $\tau_\psi(\phi) = \phi$. The intersection $\phi \cap \psi$ contains a common generator of ϕ and ψ, namely $\langle \boldsymbol{n}, \boldsymbol{c}_1 \rangle$, thus

$$\phi \cap \psi = \langle \boldsymbol{n}, \boldsymbol{c}_1 \rangle \cup c,$$

where c is a twisted cubic meeting $\langle \boldsymbol{n}, \boldsymbol{c}_1 \rangle$ twice. The curve c is invariant under τ_ψ because ϕ and ψ are both invariant under τ_ψ. The projection of c from $\boldsymbol{n}$ is a conic, because c lies in ϕ. It follows that $\boldsymbol{n}$ is a point of c. The curve c also contains $\boldsymbol{i}_n$ and $\boldsymbol{j}_n$, thus by theorem 6.2, c is a horopter curve. Hence c is in $\mathscr{H}_1(\boldsymbol{n})$, as required.

It has thus been shown that $\mathscr{H}_1(\boldsymbol{n})$ is a one-parameter family of horopter curves parametrized by $g_2(\boldsymbol{p})$. Similar arguments establish that $\mathscr{H}_2(\boldsymbol{n})$ and the $\mathscr{H}_i(\boldsymbol{m})$ are also one-parameter families of horopter curves. □

6.4. *Conjugacy classes of horopter curves*

It is shown in §3 that if c_1 and c_2 are any two non-singular twisted cubics then there exists a collineation ω of $\mathbb{P}^3$ such that $\omega(c_1) = c_2$. As a consequence, the projective geometric properties of c_1 and c_2 are the same. If c_1 and c_2 are non-singular horopter curves then it is natural to ask if there exists a euclidean transformation ω of $\mathbb{P}^3$ such that $\omega(c_1) = c_2$. In contrast with the more general case, an appropriate euclidean transformation does not always exist. Instead, there exists a one-parameter family of horopter curves c_α such that a given c_α cannot be transformed to any other member of the family by a euclidean transformation. The family c_α is complete, in that an arbitrary non-singular horopter curve c is of the form $c = \omega(c_\alpha)$, for some choice of ω and α. A suitable family c_α is obtained in this section and shown to have the required properties. It is convenient to make the following definition.

Definition 6.3. Two horopter curves, c_1 and c_2 are defined to be conjugate if there exists a euclidean transformation ω of $\mathbb{P}^3$ such that $\omega(c_1) = c_2$. Conjugacy between horopter curves is indicated by $c_1 \sim c_2$.

It follows from definition 6.3 that conjugacy of horopter curves is an equivalence relation: if c_1, c_2, c_3 are three horopter curves, then (i) $c_1 \sim c_1$; (ii) if $c_1 \sim c_2$, then $c_2 \sim c_1$; and (iii) if $c_1 \sim c_2$ and $c_2 \sim c_3$, then $c_1 \sim c_3$. Consequently, the set of horopter curves is partitioned into disjoint equivalence classes, such that the horopter curves in a given class are conjugate to one another, and such that two horopter curves from different classes are not conjugate.

The following theorem is a necessary preliminary.

Theorem 6.9. *Let ψ be a non-singular rectangular quadric with a principal point $\boldsymbol{n}$. Then the horopter curves in $\mathscr{H}_1(\boldsymbol{n})$ and $\mathscr{H}_2(\boldsymbol{n})$ all have the same real asymptotic line.*

Proof. Let l be the real asymptotic line of a horopter curve c in $\mathscr{H}_1(\boldsymbol{n})$ or $\mathscr{H}_2(\boldsymbol{n})$. Then l is in the tangent plane Π to ψ at $\boldsymbol{n}$. The line l is invariant under the rigid skew involution τ_ψ of ψ, thus l is a common transversal of the lines h_τ, g_τ of fixed points of τ_ψ. Let h_τ be the line of fixed points of τ_ψ not contained in Π_∞. Then $l = \langle \boldsymbol{n}, h_\tau \cap \Pi \rangle$, thus l is independent of the choice of c. □

Theorem 6.10. *The horopter curves in a single family $\mathscr{H}_i(\boldsymbol{m})$ or $\mathscr{H}_i(\boldsymbol{n})$ are pairwise non-conjugate.*

Proof. Let c_1, c_2 be distinct horopter curves in $\mathscr{H}_1(\boldsymbol{n})$, and let ω be a euclidean transformation such that $\omega(c_1) = c_2$. It is thus required to prove that $c_1 = c_2$.

Let h_τ, g_τ be the two skew lines of fixed points of τ_ψ, chosen such that g_τ is contained in Π_∞. The collineation $\omega^{-1}\tau_\psi\omega$ is a non-trivial rigid skew involution of c_2 thus, by theorem 6.4, $\omega^{-1}\tau_\psi\omega = \tau_\psi$. Let $\boldsymbol{p}$ be any point of h_τ. Then $\omega^{-1}\tau_\psi\omega\boldsymbol{p} = \boldsymbol{p}$, thus $\omega\boldsymbol{p}$ is a fixed point of τ_ψ. It follows that either $\omega(h_\tau) = h_\tau$ or $\omega(h_\tau) = g_\tau$. As Π_∞ is invariant under ω, $\omega(h_\tau) = h_\tau$. Similarly, $\omega(g_\tau) = g_\tau$. It follows that the point $h_\tau \cap \Pi_\infty$ is fixed by ω. The point $\boldsymbol{n}$ is also fixed by ω because $\boldsymbol{n}$ is the only point of

$$c_1 \cap \Pi_\infty = c_2 \cap \Pi_\infty = \{\boldsymbol{n}, \boldsymbol{i}_n, \boldsymbol{j}_n\}$$

not contained in Ω. On composing ω with τ_ψ if necessary, it can be assumed that $\boldsymbol{i}_n$ and $\boldsymbol{j}_n$ are fixed by ω. In consequence, the four points $\{h_\tau \cap \Pi_\infty, \boldsymbol{n}, \boldsymbol{i}_n, \boldsymbol{j}_n\}$ are fixed by ω, thus the restriction of ω to Π_∞ is the identity.

The common centre $\boldsymbol{c}$ of c_1 and c_2 is the unique point at which c_1 and c_2 meet h_τ. The line h_τ is invariant under ω, thus $\omega\boldsymbol{c} = \boldsymbol{c}$. It follows from theorem 6.9 that c_1 and

c_2 have the same real asymptotic line l. The line l is invariant under ω, thus the intersection $l \cap h_\tau$ is a fixed point of ω. The line h_τ thus contains three fixed points of ω. Hence, the restriction of ω to h_τ is the identity. It follows that ω is the identity. □

It follows from theorem 6.10 that one-parameter families of pairwise non-conjugate horopter curves exist, namely, $\mathscr{H}_i(\boldsymbol{m})$, $\mathscr{H}_i(\boldsymbol{n})$. It remains only to show that an arbitrary horopter curve is conjugate to a curve in $\mathscr{H}_i(\boldsymbol{m})$ or $\mathscr{H}_i(\boldsymbol{n})$.

Theorem 6.11. *A non-singular horopter curve is conjugate to one of the horopter curves contained in the fixed non-singular quadric ψ and invariant under τ_ψ.*

Proof. It is shown that an arbitrary non-singular horopter curve c is conjugate to a horopter curve in $\mathscr{H}_1(\boldsymbol{n})$. It follows from the proof of theorem 6.8 that the horopter curves in $\mathscr{H}_1(\boldsymbol{n})$ have a common centre $\boldsymbol{c}_1$, and it follows from theorem 6.9 that the horopter curves in $\mathscr{H}_1(\boldsymbol{n})$ have a common real asymptotic line l_n.

Let c be invariant under the rigid skew involution τ_c, and let g_τ, h_τ be the lines of fixed points of τ_c, labelled such that g_τ is contained in Π_∞. Let $\boldsymbol{c}$ be the centre of c, and let l be the real asymptotic line of c. Let l_ψ be the axis of ψ, and let l_n be the common real asymptotic line of the horopter curves in $\mathscr{H}_1(\boldsymbol{n})$. Then there exists a euclidean transformation ω such that

$$\omega(g_\tau) = \langle \boldsymbol{m}, \boldsymbol{n} \rangle, \quad \omega(h_\tau) = l_\psi, \quad \omega(l) = l_n, \quad \omega(\boldsymbol{c}) = \boldsymbol{c}_1.$$

Let $c' = \omega(c)$. Then c' is invariant under the skew involution $\omega\tau_c\omega^{-1} = \tau_\psi$.

It remains to show that c' is contained in $\mathscr{H}_1(\boldsymbol{n})$. Let Π be the tangent plane to ψ at $\boldsymbol{c}_1$, and let k be the tangent line of c' at $\boldsymbol{c}_1$. The line k is invariant under τ_ψ, thus k is contained in Π. Each line tangent to ψ at $\boldsymbol{c}_1$ is also the tangent line to a horopter curve in $\mathscr{H}_1(\boldsymbol{n})$. It follows that there exists a curve c'' in $\mathscr{H}_1(\boldsymbol{n})$ with tangent line k at $\boldsymbol{c}_1$. It follows from theorem 6.7 that $c' = c''$, thus c is conjugate to a horopter curve in $\mathscr{H}_1(\boldsymbol{n})$. □

6.5. *Examples of twisted cubics and horopter curves*

Some explicit parametrizations of twisted cubics and horopter curves are obtained. The twisted cubics are required to be invariant under the non-trivial rigid skew involution τ defined by

$$\tau = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

It is also required that the twisted cubics pass through fixed points $\boldsymbol{c}$, $\boldsymbol{n}$ of τ given by

$$\boldsymbol{c} = (0,0,0,1)^{\mathrm{T}}, \quad \boldsymbol{n} = (1,0,0,0)^{\mathrm{T}}.$$

Let c be a twisted cubic with a parametrization $t \mapsto (f(t), g(t), h(t), k(t))$ where

$$\left.\begin{aligned} f(t) &= f_0 + f_1 t + f_2 t^2 + f_3 t^3, \\ g(t) &= g_0 + g_1 t + g_2 t^2 + g_3 t^3, \\ h(t) &= h_0 + h_1 t + h_2 t^2 + h_3 t^3, \\ k(t) &= k_0 + k_1 t + k_2 t^2 + k_3 t^3. \end{aligned}\right\} \tag{36}$$

The curve c can be parametrized by t in many different ways. To fix a unique parametrization of c, it is required that the point $\boldsymbol{c}$ corresponds to $t = 0$, the point $\boldsymbol{n}$

corresponds to $t = \infty$, and that one of the intersections of c with Π_∞ corresponds to $t = \mathrm{i}$. These requirements ensure that the values of t at the remaining points of c are uniquely determined.

With the above choice of parametrization, the restriction of τ to c is given by $t \mapsto -t$, because this is the only involution of c with fixed points at $t = 0$ and $t = \infty$. The three points of $c \cap \Pi_\infty$ are thus $t = 0$, $t = \mathrm{i}$, $t = -\mathrm{i}$. It should be noted that the points $t = \pm\mathrm{i}$ need not be the points of contact of the tangents drawn from $\boldsymbol{n}$ to the absolute conic.

As a result of the above restrictions on c, and the choice of parametrization of c, the equations of (36) reduce to

$$\left.\begin{aligned} f(t) &= f_1 t + f_3 t^3, \quad & g(t) &= g_1 t + g_2 t^2,\\ h(t) &= g_1 t - g_2 t^2, \quad & k(t) &= k_0(1+t^2).\end{aligned}\right\} \tag{37}$$

It follows from (37) that the points $t = \pm\mathrm{i}$ at which c meets Π_∞ are given by

$$(\mathrm{i}(f_1 - f_3), \mathrm{i}g_1 - g_2, \mathrm{i}g_1 + g_2, 0)^{\mathrm{T}},$$
$$(-\mathrm{i}(f_1 - f_3), -\mathrm{i}g_1 - g_2, -\mathrm{i}g_1 + g_2, 0)^{\mathrm{T}}.$$

In general, neither of these points is equal to the points of contact $\boldsymbol{i}_n$, $\boldsymbol{j}_n$ of the tangents drawn from $\boldsymbol{n}$ to Ω, thus the twisted cubics of (37) are not, in general, horopter curves.

Let c now be a horopter curve, and let $t = \mathrm{i}$ correspond to the point $\boldsymbol{i}_n = (0, 1, \mathrm{i}, 0)^{\mathrm{T}}$. It follows that

$$(\mathrm{i}(f_1 - f_3), \mathrm{i}g_1 - g_2, \mathrm{i}g_1 + g_2, 0)^{\mathrm{T}} = (0, 1, \mathrm{i}, 0)^{\mathrm{T}},$$

thus $f_1 = f_3$ and $g_1 = -g_2$. The equations of (37) reduce to

$$\left.\begin{aligned} f(t) &= f_1(t + t^3), \quad & g(t) &= g_1(t - t^2),\\ h(t) &= g_1(t + t^2), \quad & k(t) &= k_0(1+t^2).\end{aligned}\right\} \tag{38}$$

The coefficients f_1, g_1, k_0 of (38) can be further restricted by specifying the tangent line to c at $\boldsymbol{n}$. For example, if this line is required to be $\langle \boldsymbol{n}, (0, -1, 1, 1)^{\mathrm{T}}\rangle$, then $k_0 = g_1$.

7. Ambiguous surfaces

The results obtained in §5 and §6 are applied to ambiguous surfaces. The notation used in §2 is recalled. The origin $\boldsymbol{o}$ is taken as the optical centre for the first image and $\boldsymbol{a}$, $\boldsymbol{b}$ are taken as the two possible optical centres for the second image. The ambiguous surface ψ is the result of reconstruction when the second camera has its optical centre at $\boldsymbol{a}$, and the complementary ambiguous surface ϕ is the result of reconstruction when the second camera has its optical centre at $\boldsymbol{b}$. The surfaces ψ, ϕ each contain $\boldsymbol{o}$, $\boldsymbol{a}$, $\boldsymbol{b}$. In addition, ψ contains $\langle \boldsymbol{o}, \boldsymbol{b}\rangle$, and ϕ contains $\langle \boldsymbol{o}, \boldsymbol{a}\rangle$. The main new result of this section is a cubic polynomial constraint on ambiguous surfaces obtained from τ_ψ.

7.1. *Rigid involutions of ambiguous surfaces*

The following theorem establishes a link between rigid involutions of ambiguous surfaces and rigid involutions of horopter curves.

Theorem 7.1. *The ambiguous surface complementary to a given non-singular ambiguous surface can be chosen such that both the horopter curve contained in the intersection of the two ambiguous surfaces and the given ambiguous surface are invariant under the same non-trivial rigid involution.*

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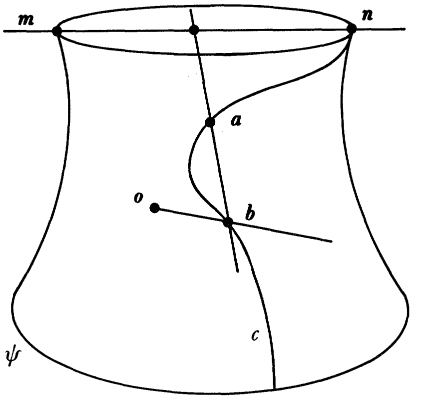

Figure 13. Illustration to theorem 7.1.

Proof. Let ψ be a non-singular ambiguous surface with equation

$$(U\boldsymbol{x}\times\boldsymbol{x})\cdot\boldsymbol{b} = (U\boldsymbol{a}\times\boldsymbol{x})\cdot\boldsymbol{b} \tag{39}$$

identical to (7). It follows from (39) that the optical centre $\boldsymbol{b}$ of the camera taking the second image lies on a generator g of ψ passing through $\boldsymbol{o}$. The position of $\boldsymbol{b}$ is not uniquely determined, in that $\boldsymbol{b}$ can be moved on g, without changing ψ. As $\boldsymbol{b}$ moves on g, the ambiguous surface, ϕ, complementary to ψ and the horopter curve contained in $\psi\cap\phi$ both change.

Let $\boldsymbol{b}$ be chosen on g such that $\langle\boldsymbol{a},\boldsymbol{b}\rangle$ meets Π_∞ at a point of $\langle\boldsymbol{m},\boldsymbol{n}\rangle$, as shown in figure 13. Let $\boldsymbol{m}$, $\boldsymbol{n}$ be the principal points of ψ, and let c be the horopter curve contained in $\psi\cap\phi$. It follows from theorem 6.3 that c is invariant under a non-trivial rigid skew involution τ. Let g_τ and h_τ be the skew lines of fixed points of τ, labelled such that g_τ is contained in Π_∞. It follows from the explicit construction of τ, given in theorem 6.3, that g_τ contains both $\boldsymbol{n}$ and $\langle\boldsymbol{a},\boldsymbol{b}\rangle\cap\langle\boldsymbol{m},\boldsymbol{n}\rangle$, thus $g_\tau = \langle\boldsymbol{m},\boldsymbol{n}\rangle$.

Let $s_\infty = \psi\cap\Pi_\infty$. The intersection $\tau(s_\infty)\cap s_\infty$ contains the six points $\boldsymbol{m}$, $\boldsymbol{i}_m$, $\boldsymbol{j}_m$, $\boldsymbol{n}$, $\boldsymbol{i}_n$, $\boldsymbol{j}_n$, thus $\tau(s_\infty) = s_\infty$. It follows that $\tau(\psi)\cap\psi$ contains a (split) curve of degree five comprising the cubic space curve c contained in $\psi\cap\phi$ and the conic s_∞. The intersection of two distinct quadrics is a curve of degree four, thus $\tau(\psi)$ and ψ are not distinct. In other words $\tau(\psi)=\psi$. □

Corollary. *The rigid involution τ is equal to the rigid involution τ_ψ defined in* §5.3, *because τ_ψ is the unique non-trivial rigid skew involution of ψ which fixes both $\boldsymbol{m}$ and $\boldsymbol{n}$.*

Theorem 7.1 can be proved algebraically, starting from (25) and (39). It is sufficient to show that $\tau_\psi\boldsymbol{a}\times\boldsymbol{b} = 0$. A long calculation shows that

$$(\tau_\psi\boldsymbol{a}\times\boldsymbol{b})\cdot\boldsymbol{m} = 0,$$
$$(\tau_\psi\boldsymbol{a}\times\boldsymbol{b})\cdot\boldsymbol{n} = 0,$$
$$(\tau_\psi\boldsymbol{a}\times\boldsymbol{b})\cdot(\boldsymbol{m}\times\boldsymbol{n}) = 0.$$

The result then follows. The details are omitted.

The converse of theorem 7.1 is obtained as follows.

Theorem 7.2. *With the notation of theorem* 7.1, *let $\boldsymbol{o}$, $\boldsymbol{a}$, $\tau_\psi\boldsymbol{a}$ be distinct points of the non-singular rectangular quadric ψ such that $\tau_\psi\boldsymbol{a}$ is on a generator of ψ passing through $\boldsymbol{o}$. Then ψ is an ambiguous surface such that $\boldsymbol{o}$ is the optical centre for the camera taking the first image, and $\boldsymbol{a}$, $\tau_\psi\boldsymbol{a}$ are the two possible optical centres for the camera taking the second image.*

Proof. Let $\boldsymbol{b} = \tau_\psi \boldsymbol{a}$. It follows from the proof of theorem 6.8 that ψ contains a unique horopter curve c invariant under τ_ψ and containing $\boldsymbol{n}$ and $\boldsymbol{a}$, such that c intersects the generator $\langle \boldsymbol{o}, \boldsymbol{b} \rangle$ exactly once. The curve c contains $\boldsymbol{b} = \tau_\psi \boldsymbol{a}$, thus $c \cap \langle \boldsymbol{o}, \boldsymbol{b} \rangle = \boldsymbol{b}$. It follows from theorem 6.5 that there exists an orthogonal collineation ω from the star of lines through $\boldsymbol{a}$ to the star of lines through $\boldsymbol{b}$ such that c is the locus of the intersections $k \cap \omega k$ as k varies through the lines containing $\boldsymbol{b}$.

Let ψ' be the surface swept out by the lines $\Pi \cap \omega(\Pi)$ as Π varies through the pencil of planes containing $\omega^{-1}\langle \boldsymbol{o}, \boldsymbol{b} \rangle$. It follows from the construction given in §2.2 that ψ' is an ambiguous surface. The intersection, $\psi \cap \psi'$, contains the horopter curve c, and the line $\langle \boldsymbol{o}, \boldsymbol{b} \rangle$. In addition, $\psi \cap \psi'$ contains a line h which meets c twice, because this is a general property of a twisted cubic contained in the intersection of two quadric surfaces (Semple & Kneebone 1952). Now $h \neq \langle \boldsymbol{o}, \boldsymbol{b} \rangle$, because $\langle \boldsymbol{o}, \boldsymbol{b} \rangle$ meets c only once. It follows that $\psi \cap \psi'$ contains a (split) space curve of degree five, namely $c \cup \langle \boldsymbol{o}, \boldsymbol{b} \rangle \cup h$. Two distinct quadrics intersect in a space curve of degree four, thus ψ and ψ' are not distinct. In other words, $\psi = \psi'$. □

7.2. *An expression for* τ_ψ

An explicit algebraic expression is obtained for the rigid skew involution τ_ψ appearing in theorem 7.1. With the notation of theorem 7.1, let the principal points of ψ be $\boldsymbol{m}$, $\boldsymbol{n}$, and let cartesian coordinates be chosen such that ψ has the equation

$$\boldsymbol{x}^{\mathrm{T}} M \boldsymbol{x} + \boldsymbol{l} \cdot \boldsymbol{x} = 0, \tag{40}$$

where $\boldsymbol{x} = [x_1, x_2, x_3]$ is a point of $\mathbb{R}^3$, $\boldsymbol{l}$ is a vector, and M is a real 3×3 symmetric matrix of the form

$$M = \tfrac{1}{2}(\boldsymbol{m} \otimes \boldsymbol{n} + \boldsymbol{n} \otimes \boldsymbol{m}) - \boldsymbol{m} \cdot \boldsymbol{n} I. \tag{41}$$

The tangent planes to ψ at $\boldsymbol{m}$ and $\boldsymbol{n}$ are given by

$$2\boldsymbol{m}^{\mathrm{T}} M \boldsymbol{x} + \boldsymbol{m} \cdot \boldsymbol{l} = 0, \tag{42}$$

$$2\boldsymbol{n}^{\mathrm{T}} M \boldsymbol{x} + \boldsymbol{n} \cdot \boldsymbol{l} = 0. \tag{43}$$

It follows from the definition of τ_ψ given in §5.3 that the axis of τ_ψ is the line l_ψ formed by the intersection of the two planes (42) and (43). By theorem 5.3, l_ψ meets Π_∞ at $\boldsymbol{m} \times \boldsymbol{n}$, thus l_ψ has a parametrization of the form

$$t \mapsto \boldsymbol{s} + t\boldsymbol{m} \times \boldsymbol{n},$$

where

$$\boldsymbol{s} \cdot (\boldsymbol{m} \times \boldsymbol{n}) = 0. \tag{44}$$

It follows from (42), (43) and (44) that l_ψ is given by

$$t \mapsto -[(\boldsymbol{n} \cdot \boldsymbol{l})\,\boldsymbol{m} + (\boldsymbol{m} \cdot \boldsymbol{l})\,\boldsymbol{n}]/\|\boldsymbol{m} \times \boldsymbol{n}\|^2 + t\boldsymbol{m} \times \boldsymbol{n}. \tag{45}$$

Let $\boldsymbol{x}$ be any point of $\mathbb{R}^3$. The mid-point of the line segment $[\boldsymbol{x}, \tau_\psi \boldsymbol{x}]$ is on l_ψ, thus there exists a value of t, depending on $\boldsymbol{x}$, such that

$$\tfrac{1}{2}[\boldsymbol{x} + \tau_\psi \boldsymbol{x}] = \boldsymbol{s} + t(\boldsymbol{m} \times \boldsymbol{n}). \tag{46}$$

The line $\langle \boldsymbol{x}, \tau_\psi \boldsymbol{x} \rangle$ is orthogonal to l_ψ thus

$$(\boldsymbol{x} - \tau_\psi \boldsymbol{x}) \cdot (\boldsymbol{m} \times \boldsymbol{n}) = 0. \tag{47}$$

It follows from (46) and (47) that

$$t = \boldsymbol{x} \cdot (\boldsymbol{m} \times \boldsymbol{n})/\|\boldsymbol{m} \times \boldsymbol{n}\|^2.$$

On substituting for t in (46) the following expression for $\tau_\psi \boldsymbol{x}$ is obtained.

$$\tau_\psi \boldsymbol{x} = -\frac{2[(\boldsymbol{n}\cdot\boldsymbol{l})\,\boldsymbol{m}+(\boldsymbol{m}\cdot\boldsymbol{l})\,\boldsymbol{n}]}{\|\boldsymbol{m}\times\boldsymbol{n}\|^2}+\frac{2[\boldsymbol{x}\cdot(\boldsymbol{m}\times\boldsymbol{n})]\,\boldsymbol{m}\times\boldsymbol{n}}{\|\boldsymbol{m}\times\boldsymbol{n}\|^2}-\boldsymbol{x}. \tag{48}$$

7.3. *Two cubic constraints on ambiguous surfaces*

Two cubic constraints on ambiguous surfaces are obtained. Three theorems are required to derive the first constraint. The second constraint is obtained more easily.

Theorem 7.3. *With the notation of theorem* 7.1, *let* ψ *be a non-singular ambiguous surface with principal points* $\boldsymbol{m}$, $\boldsymbol{n}$, *and let cartesian coordinates be chosen such that the origin* $\boldsymbol{o}$ *is the optical centre of the camera taking the first image. Let* $\boldsymbol{r} = \boldsymbol{m}\times\boldsymbol{n}$, *let* $\boldsymbol{l}$ *be the normal to the tangent plane to* ψ *at* $\boldsymbol{o}$, *and let* $\boldsymbol{a}$ *be a possible optical centre for the second camera not lying on a generator through* $\boldsymbol{o}$. *Then*

$$-4(\boldsymbol{m}\cdot\boldsymbol{l})(\boldsymbol{n}\cdot\boldsymbol{l})+2(\boldsymbol{a}\cdot\boldsymbol{r})(\boldsymbol{l}\cdot\boldsymbol{r})-(\boldsymbol{a}\cdot\boldsymbol{l})(\boldsymbol{r}\cdot\boldsymbol{r}) = 0. \tag{49}$$

Proof. Let τ_ψ be the unique non-trivial rigid involution of ψ that fixes both $\boldsymbol{m}$ and $\boldsymbol{n}$. It follows from theorem 7.1 that $\tau_\psi \boldsymbol{a}$ lies in the tangent plane to ψ at $\boldsymbol{o}$, thus $\boldsymbol{l}\cdot\tau_\psi \boldsymbol{a} = 0$. The result follows on substituting the expression for $\tau_\psi \boldsymbol{a}$ given by (48) into the equation $\boldsymbol{l}\cdot\tau_\psi \boldsymbol{a} = 0$. □

Theorem 7.4. *Let* $\boldsymbol{m}$, $\boldsymbol{n}$, $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$ *be vectors such that*

$$\tfrac{1}{2}(\boldsymbol{m}\otimes\boldsymbol{n}+\boldsymbol{n}\otimes\boldsymbol{m}) = \begin{bmatrix}\boldsymbol{e}_1^{\mathrm{T}}\\ \boldsymbol{e}_2^{\mathrm{T}}\\ \boldsymbol{e}_3^{\mathrm{T}}\end{bmatrix},$$

then

$$\tfrac{1}{4}(\boldsymbol{m}\times\boldsymbol{n})\otimes(\boldsymbol{m}\times\boldsymbol{n}) = \begin{bmatrix}\boldsymbol{e}_3^{\mathrm{T}}\times\boldsymbol{e}_2^{\mathrm{T}}\\ \boldsymbol{e}_1^{\mathrm{T}}\times\boldsymbol{e}_3^{\mathrm{T}}\\ \boldsymbol{e}_2^{\mathrm{T}}\times\boldsymbol{e}_1^{\mathrm{T}}\end{bmatrix}. \tag{50}$$

Proof. It follows from the hypothesis of the theorem that $2\boldsymbol{e}_i = m_i\boldsymbol{n}+n_i\boldsymbol{m}$. The result follows on substituting for $\boldsymbol{e}_i$ in (50). □

The first cubic polynomial constraint on ψ is now obtained.

Theorem 7.5. *In the reconstruction problem, let two points in space be given as optical centres for the cameras taking the first and second images respectively. Then any ambiguous surface reconstructed from corresponding points in the two images satisfies a cubic polynomial constraint.*

Proof. Cartesian coordinates are chosen, and the notation of theorem 7.1 is used. It follows from theorem 5.4 that the equation for ψ is

$$\boldsymbol{x}^{\mathrm{T}}M\boldsymbol{x}+\boldsymbol{l}\cdot\boldsymbol{x} = 0,$$

where M, $\boldsymbol{l}$ are defined as in (40) and (41). Define the matrices N, L by

$$N = \tfrac{1}{2}(\boldsymbol{m}\otimes\boldsymbol{n}+\boldsymbol{n}\otimes\boldsymbol{m}), \tag{51}$$

$$L = (\boldsymbol{m}\times\boldsymbol{n})\otimes(\boldsymbol{m}\times\boldsymbol{n}). \tag{52}$$

It follows from (41) and (51) that $N = M-\frac{1}{2}\mathrm{tr}\,(M)\,I$. The entries of N are thus linear functions of the entries of M. It follows from theorem 7.4 that the entries of L are quadratic functions of the entries of M.

A cubic polynomial constraint on ψ is obtained from (49). The term $-4(\boldsymbol{l}\cdot\boldsymbol{m})(\boldsymbol{l}\cdot\boldsymbol{n})$ on the left-hand side of (49) has the form

$$-4(\boldsymbol{l}\cdot\boldsymbol{m})(\boldsymbol{l}\cdot\boldsymbol{n}) = -4\boldsymbol{l}^{\mathrm{T}}N\boldsymbol{l}. \tag{53}$$

The remaining two terms on the left-hand side of (49) have the form

$$2(\boldsymbol{a}\cdot\boldsymbol{r})(\boldsymbol{l}\cdot\boldsymbol{r})-(\boldsymbol{a}\cdot\boldsymbol{l})(\boldsymbol{r}\cdot\boldsymbol{r}) = 2\boldsymbol{a}^{\mathrm{T}}L\boldsymbol{l}-(\boldsymbol{a}\cdot\boldsymbol{l})\,\mathrm{tr}\,(L). \tag{54}$$

It follows from (53) and (54) that (49) is equivalent to the cubic constraint

$$-4\boldsymbol{l}^{\mathrm{T}}N\boldsymbol{l}+2\boldsymbol{a}^{\mathrm{T}}L\boldsymbol{l}-(\boldsymbol{a}\cdot\boldsymbol{l})\,\mathrm{tr}\,(L) = 0. \tag{55}$$

□

Corollary. Let $\boldsymbol{e}_1^{\mathrm{T}}, \boldsymbol{e}_2^{\mathrm{T}}, \boldsymbol{e}_3^{\mathrm{T}}$ be the rows of N. It follows from theorem 7.5 and (52) that (55) is equivalent to

$$-\boldsymbol{l}^{\mathrm{T}}\begin{bmatrix}\boldsymbol{e}_1^{\mathrm{T}}\\ \boldsymbol{e}_2^{\mathrm{T}}\\ \boldsymbol{e}_3^{\mathrm{T}}\end{bmatrix}\boldsymbol{l}+2\boldsymbol{a}^{\mathrm{T}}\begin{bmatrix}\boldsymbol{e}_3^{\mathrm{T}}\times\boldsymbol{e}_2^{\mathrm{T}}\\ \boldsymbol{e}_1^{\mathrm{T}}\times\boldsymbol{e}_3^{\mathrm{T}}\\ \boldsymbol{e}_2^{\mathrm{T}}\times\boldsymbol{e}_1^{\mathrm{T}}\end{bmatrix}\boldsymbol{l}-(\boldsymbol{a}\cdot\boldsymbol{l})\,\mathrm{tr}\begin{bmatrix}\boldsymbol{e}_3^{\mathrm{T}}\times\boldsymbol{e}_2^{\mathrm{T}}\\ \boldsymbol{e}_1^{\mathrm{T}}\times\boldsymbol{e}_3^{\mathrm{T}}\\ \boldsymbol{e}_2^{\mathrm{T}}\times\boldsymbol{e}_1^{\mathrm{T}}\end{bmatrix} = 0. \tag{56}$$

The second cubic polynomial constraint on ψ referred to at the beginning of this section is

$$\det\,(N) = 0. \tag{57}$$

Equation (57) is well known (Horn 1987). It follows from the observations (i) that N has rank two; and (ii) that N depends linearly on the coefficients of the equation defining the ambiguous surface. Equation (57) is equivalent to a simple constraint on the eigenvalues of the matrix M. To show this, (57) is first written in the form

$$\det\,[M-\tfrac{1}{2}\,\mathrm{tr}\,(M)\,I] = 0.$$

It follows that $\frac{1}{2}\,\mathrm{tr}\,(M)$ is an eigenvalue of M. Let λ_1, λ_2, λ_3 be the eigenvalues of M, labelled such that $\lambda_3 = \frac{1}{2}\,\mathrm{tr}\,(M)$. Then

$$\tfrac{1}{2}\,\mathrm{tr}\,(M) = \tfrac{1}{2}(\lambda_1+\lambda_2+\lambda_3) = \lambda_3.$$

Thus (57) is equivalent to $\lambda_1+\lambda_2 = \lambda_3$.

7.4. *A special case*

The cubic constraints of §7.3 are applied to a particular one-dimensional space of quadrics on which they take a particularly simple form. The space of quadrics is constructed such that all the quadrics in the space satisfy (57), and such that within this space (56) has a simple geometric interpretation.

Let ψ_1, ψ_2 be two non-singular rectangular quadrics with principal points $\boldsymbol{m}$, $\boldsymbol{n}_1$ and $\boldsymbol{m}$, $\boldsymbol{n}_2$ respectively, such that $\boldsymbol{m}$, $\boldsymbol{n}_1$, $\boldsymbol{n}_2$ lie on a fixed line g, and let $\boldsymbol{o}$, $\boldsymbol{a}$ be distinct real points of $\mathbb{R}^3$ contained in both ψ_1 and ψ_2. Let $\langle\psi_1,\psi_2\rangle$ be the one-dimensional space of quadrics generated by ψ_1 and ψ_2. A general real quadric ψ in $\langle\psi_1,\psi_2\rangle$ has the form

$$\psi = \lambda_1\psi_1+\lambda_2\psi_2,$$

where (λ_1,λ_2) is a real point of $\mathbb{P}^1$. The quadric ψ meets Π_∞ in the conic $\boldsymbol{x}^{\mathrm{T}}M\boldsymbol{x} = 0$ where

$$M = \tfrac{1}{2}[\boldsymbol{m}\otimes(\lambda_1\boldsymbol{n}_1+\lambda_2\boldsymbol{n}_2)+(\lambda_1\boldsymbol{n}_1+\lambda_2\boldsymbol{n}_2)\otimes\boldsymbol{m}]-\boldsymbol{m}\cdot(\lambda_1\boldsymbol{n}_1+\lambda_2\boldsymbol{n}_2)\,I.$$

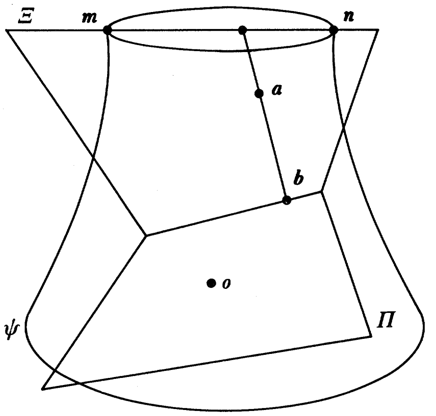

Figure 14. The constraint of (56) applied to a pencil of rectangular quadrics.

The quadric ψ thus satisfies (57) for all choices of (λ_1, λ_2). The principal points of ψ are $\boldsymbol{m}$, $\boldsymbol{n}$ where $\boldsymbol{n}$ is the point of g given by $\boldsymbol{n} = \lambda_1 \boldsymbol{n}_1 + \lambda_2 \boldsymbol{n}_2$.

The quadric ψ contains $\boldsymbol{o}$ and $\boldsymbol{a}$, since ψ_1 and ψ_2 both contain $\boldsymbol{o}$ and $\boldsymbol{a}$. Let τ_ψ be the unique non-trivial rigid involution of ψ that fixes both $\boldsymbol{m}$ and $\boldsymbol{n}$. Then g is one of the lines of fixed points of τ_ψ. It follows that the plane $\Xi = \langle g, \boldsymbol{a} \rangle$ is invariant under τ_ψ. The geometry associated with ψ is illustrated in figure 14.

Let $\boldsymbol{b} = \tau_\psi \boldsymbol{a}$. Then $\boldsymbol{b}$ is in Ξ, because $\boldsymbol{a}$ is in Ξ, and Ξ is invariant under τ_ψ. It follows from (48) that

$$\boldsymbol{b} = -\frac{2[(\boldsymbol{n}\cdot\boldsymbol{l})\,\boldsymbol{m} + (\boldsymbol{m}\cdot\boldsymbol{l})\,\boldsymbol{n}]}{\|\boldsymbol{m}\times\boldsymbol{n}\|^2} + \frac{2[\boldsymbol{a}\cdot(\boldsymbol{m}\times\boldsymbol{n})]\,\boldsymbol{m}\times\boldsymbol{n}}{\|\boldsymbol{m}\times\boldsymbol{n}\|^2} - \boldsymbol{a}.$$

Let N, L be the matrices defined by (51) and (52), respectively. Then

$$\boldsymbol{b} = (-4N\boldsymbol{l} + 2L\boldsymbol{a})/\mathrm{tr}\,(L) - \boldsymbol{a}. \tag{58}$$

In (58), $\boldsymbol{b}$ is regarded as a point of $\mathbb{R}^3$. It is convenient to regard $\boldsymbol{b}$ as a point of $\mathbb{P}^3$. The projective coordinates of $\boldsymbol{b}$ are

$$\boldsymbol{b} = (-4N\boldsymbol{l} + 2L\boldsymbol{a} - \mathrm{tr}\,(L)\,\boldsymbol{a}, \mathrm{tr}\,(L))^{\mathrm{T}}.$$

The matrix N and the vector $\boldsymbol{l}$ are linear functions of ψ (or equivalently, linear functions of (λ_1, λ_2)). The matrix L is a quadratic function of ψ. Thus $\boldsymbol{b}$ is a quadratic function of ψ. Thus, as ψ varies, the locus of $\boldsymbol{b}$ is a conic in Ξ. Let this conic be s, and let $\boldsymbol{r}$, $\boldsymbol{s}$ be two distinct fixed points on s. Let $\boldsymbol{g}_\psi$ be the coefficients of $\langle \boldsymbol{b}, \boldsymbol{r} \rangle$. By definition, a point $\boldsymbol{x}$ of Ξ is on $\langle \boldsymbol{b}, \boldsymbol{r} \rangle$ if and only if $\boldsymbol{g}_\psi \cdot \boldsymbol{x} = 0$. Similarly, let $\boldsymbol{h}_\psi$ be the coefficients of $\langle \boldsymbol{b}, \boldsymbol{s} \rangle$. It follows from Steiner's theorem that $\boldsymbol{g}_\psi$ and $\boldsymbol{h}_\psi$ are both linear functions of ψ.

Let Π be the tangent plane to ψ at $\boldsymbol{o}$. Then (56) is satisfied if and only if $\boldsymbol{b}$ lies on $\Pi \cap \Xi$. Let $\boldsymbol{k}_\psi$ be the coefficients of the line $\Pi \cap \Xi$. The plane Π depends linearly on ψ, and Ξ is fixed independently of ψ, thus $\boldsymbol{k}_\psi$ depends linearly on ψ. The condition that $\boldsymbol{b}$ lies on both s and on the line $\boldsymbol{k}_\psi \cdot \boldsymbol{x} = 0$ is

$$\det \begin{bmatrix} \boldsymbol{g}_\psi \\ \boldsymbol{h}_\psi \\ \boldsymbol{k}_\psi \end{bmatrix} = 0,$$

which is the cubic condition on ψ equivalent to (55) in this special case.

8. Five image correspondences

If only a small number of image correspondences are available then ambiguity is more likely because the data place fewer constraints on the reconstruction. If the number of image correspondences is reduced to five, then ambiguity is certain. Each image correspondence $\boldsymbol{q} \leftrightarrow \boldsymbol{q}'$ places one constraint on the camera displacement $\{R, \boldsymbol{a}\}$ of the form

$$\boldsymbol{q}' \cdot (R\boldsymbol{q} \times R\boldsymbol{a}) = 0.$$

The number of unknown parameters is five, comprising three for the rotation R, and two for the direction of the translation $\boldsymbol{a}$. Let N be the number of essentially different rigid displacements compatible with five image correspondences. For a general choice of five image correspondences, N is constant, finite and not equal to zero. The number N is an algebraic measure of the complexity of reconstruction. It is analogous to the degree of an algebraic curve. Demazure (1988) uses algebraic geometry to prove that $N = 10$. In this context ten is high, indicating that the reconstruction problem is difficult. The following four theorems comprise a new proof of Demazure's result.

Theorem 8.1. *Let five image correspondences be given compatible with a given camera displacement. Then a two-dimensional space of quadrics can be constructed such that any ambiguous surface compatible with the five image correspondences and compatible with the given camera displacement is represented by a point in the two-dimensional space of quadrics.*

Proof. Let cartesian coordinates be chosen with origin $\boldsymbol{o}$ at the optical centre of the camera from which the first image is obtained. Let $\boldsymbol{q}_i \leftrightarrow \boldsymbol{q}_i'$ be five image correspondences compatible with the given camera displacement $\{R, \boldsymbol{a}\}$, where R is an orthogonal matrix and $\boldsymbol{a}$ is the optical centre of the camera from which the second image is obtained. Let $\boldsymbol{p}_i$ be the points in $\mathbb{P}^3$ such that the image of $\boldsymbol{p}_i$ from $\boldsymbol{o}$ is $\boldsymbol{q}_i$, and the image of $\boldsymbol{p}_i$ from $\boldsymbol{a}$ is $\boldsymbol{q}_i'$, as illustrated in figure 15.

An ambiguous surface, compatible with $\boldsymbol{q}_i \leftrightarrow \boldsymbol{q}_i'$ and compatible with $\{R, \boldsymbol{a}\}$ contains the five points $\boldsymbol{p}_i$, together with the points $\boldsymbol{o}$, $\boldsymbol{a}$. The space of all quadric surfaces contained in $\mathbb{P}^3$ is of dimension nine (Semple & Kneebone 1952). The condition that a quadric contains a known point imposes a single linear condition on the quadric, thus the quadric surfaces containing the $\boldsymbol{p}_i$, $\boldsymbol{o}$ and $\boldsymbol{a}$ form a $(9-7=2)$-dimensional space S^2 within the space of all quadrics. A basis for S^2 can be calculated from the $\boldsymbol{p}_i$, $\boldsymbol{o}$ and $\boldsymbol{a}$. □

Some remarks are made on the choice of basis for the space S^2 constructed in the proof of theorem 8.1. With the notation of theorem 8.1, let cartesian coordinates be chosen such that $\boldsymbol{o} = (0,0,0,1)^{\mathrm{T}}$. Then the quadrics represented by points of S^2 are of the form

$$\boldsymbol{x}^{\mathrm{T}} M \boldsymbol{x} + \boldsymbol{l} \cdot \boldsymbol{x} = 0, \tag{59}$$

where M is a symmetric 3×3 matrix and $\boldsymbol{l}$ is a vector. The space S^2 is two dimensional (as a projective space), thus there exist symmetric matrices M_1, M_2, M_3 and corresponding vectors $\boldsymbol{l}_1$, $\boldsymbol{l}_2$, $\boldsymbol{l}_3$ such that the quadrics

$$\psi_i = \boldsymbol{x}^{\mathrm{T}} M_i \boldsymbol{x} + \boldsymbol{l}_i \cdot \boldsymbol{x}$$

span S^2. An arbitrary quadric ψ represented by a point of S^2 has an equation

$$\psi = \lambda_1 \psi_1 + \lambda_2 \psi_2 + \lambda_3 \psi_3,$$

where $(\lambda_1, \lambda_2, \lambda_3)$ is a point of $\mathbb{P}^2$ determined uniquely by ψ.

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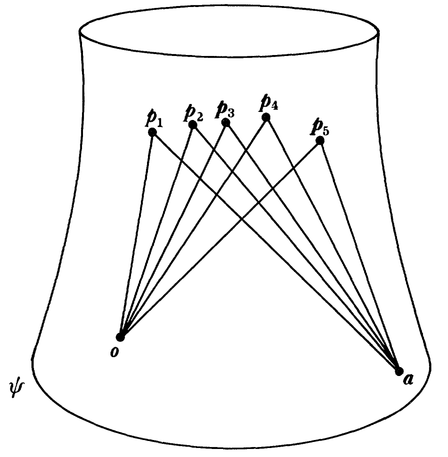

Figure 15. Seven points on a rectangular quadric.

It is not the case that all the quadrics parametrized by points of S^2 are ambiguous surfaces. The proof that there are ten essentially different camera displacements compatible with five image correspondences relies on selecting from S^2 precisely those points corresponding to ambiguous surfaces. As a preliminary, it is shown in the next theorem that each point of S^2 yields essentially only one camera displacement, different from the given camera displacement of theorem 8.1, but compatible with the image correspondences.

Theorem 8.2. *Let $\boldsymbol{p}$ be a point of the space S^2 constructed in theorem* 8.1, *such that $\boldsymbol{p}$ represents an ambiguous surface. Then $\boldsymbol{p}$ yields essentially only one rigid displacement essentially different from the rigid displacement $\{R, \boldsymbol{a}\}$ used in the construction of S^2.*

Proof. Let ψ be the ambiguous surface corresponding to $\boldsymbol{p}$. Let $\{R, \boldsymbol{a}\}$ and $\{S, \boldsymbol{b}\}$ be two essentially different rigid displacements from which ψ is obtained as specified by (7). Let τ_ψ be the unique non-trivial rigid involution of ψ that fixes the two principal points of ψ. It follows from theorem 6.1 that the generator $\langle \boldsymbol{o}, \boldsymbol{b}\rangle$ of ψ contains $\tau_\psi \boldsymbol{a}$. The direction of $\boldsymbol{b}$ is thus uniquely determined by $\boldsymbol{a}$ and τ_ψ. Hence, by definition 5.1, $\{S, \boldsymbol{b}\}$ is essentially unique. □

The ambiguous surfaces represented by points of S^2 are found as follows.

Theorem 8.3. *Let five image correspondences be given in general position. Let $\{R, \boldsymbol{a}\}$ and $\{S_i, \boldsymbol{b}_i\}$ $(1 \leqslant i \leqslant N-1)$ be a complete list of the essentially different camera displacements compatible with the image correspondences. Then each of the $N-1$ pairs $\{R, \boldsymbol{a}\}$, $(S_i, \boldsymbol{b}_i\}$ arises from an intersection point of two cubic plane curves.*

Proof. The notation of theorem 8.1 is used. Let ψ_i be the ambiguous surface constructed from $\{R, \boldsymbol{a}\}$ and $\{S_i, \boldsymbol{b}_i\}$ according to (7). It follows from theorem 8.1 that the ψ_i lie in a two-dimensional subspace S^2 of the space of all quadrics. Equations (56) and (57) yield two cubic curves f, g in S^2. Each ambiguous surface ψ_i yields an intersection point of f and g.

Conversely, let ψ be a quadric surface arising from an intersection of f and g. Then, by the definition of S^2, ψ contains the points $\boldsymbol{p}_i$ and $\boldsymbol{o}$, $\boldsymbol{a}$ and ψ is rectangular because it satisfies (57). The point $\boldsymbol{b} = \tau_\psi \boldsymbol{a}$ lies on a generator of ψ passing through $\boldsymbol{o}$, because

ψ satisfies (56). It follows from theorem 7.2 that ψ is an ambiguous surface constructed from a pair of essentially different rigid displacements $\{R, \boldsymbol{a}\}$, $(S, \boldsymbol{b}\}$ compatible with the five image correspondences. Thus $\psi = \psi_i$ for some i. □

The only remaining task is to show that the two cubic plane curves f, g in the proof of theorem 8.3 have, in general, exactly nine *distinct* intersections.

Theorem 8.4. *Let five image correspondences be given in general position. Then there are exactly ten essentially different camera displacements compatible with the given image correspondences.*

Proof. The notation of theorem 8.1 is used. It suffices to show that the cubic plane curves f, g of theorem 8.3 have, in general, exactly nine distinct intersections. To do this, it suffices to produce a single example in which the two curves have nine distinct intersections.

Let the quadrics parametrized by the points of S^2 have equations of the form (59), and let N be defined by (51). The property that two cubic plane curves have nine distinct intersections is stable against small perturbations in the coefficients of the curves, thus it is sufficient to consider the case in which the coefficients in (56) involving $\boldsymbol{a}$ are negligibly small in comparison with the first term $-\boldsymbol{l}^{\mathrm{T}}N\boldsymbol{l}$. In this case the cubic constraints (56) and (57) reduce to

$$\boldsymbol{l}^{\mathrm{T}}N\boldsymbol{l} = 0, \quad \det(N) = 0.$$

Let

$$N = \begin{bmatrix} a & d & e \\ d & b & f \\ e & f & c \end{bmatrix}$$

and let three of the reconstructed points in $\mathbb{P}^3$ be

$$\boldsymbol{p}_1 = (1, 0, 0, 0)^{\mathrm{T}}, \quad \boldsymbol{p}_2 = (0, 1, 0, 0)^{\mathrm{T}}, \quad \boldsymbol{p}_3 = (0, 0, 1, 0)^{\mathrm{T}}.$$

The $\boldsymbol{p}_i$ are in Π_∞, thus $\boldsymbol{p}_i^{\mathrm{T}} M \boldsymbol{p}_i = 0$, where M is as defined in (40). It follows that $a = b = c = 0$. Hence $M = N$, and $\det(N) = def$.

The space S^2 is parametrized by (d, e, f). The components $d = 0$, $e = 0$, $f = 0$ of $\det(N) = 0$ are considered separately. It suffices to show that, in general, each component of $\det(N) = 0$ meets the cubic plane curve $\boldsymbol{l}^{\mathrm{T}}N\boldsymbol{l} = 0$ at three distinct points. It is thus required to find two points of $\mathbb{R}^3$,

$$\boldsymbol{p}_4 = [p_{41}, p_{42}, p_{43}]^{\mathrm{T}}, \quad \boldsymbol{p}_5 = [p_{51}, p_{52}, p_{53}]^{\mathrm{T}},$$

and a vector $\boldsymbol{a}$ such that

$$\boldsymbol{p}_4^{\mathrm{T}} M \boldsymbol{p}_4 + \boldsymbol{l} \cdot \boldsymbol{p}_4 = 0, \quad \boldsymbol{p}_5^{\mathrm{T}} M \boldsymbol{p}_5 + \boldsymbol{l} \cdot \boldsymbol{p}_5 = 0, \quad \boldsymbol{l} \cdot \boldsymbol{a} = 0, \tag{60}$$

and such that the line $d = 0$ (for example) meets $\boldsymbol{l}^{\mathrm{T}}N\boldsymbol{l} = 0$ at three distinct points. The last equation of (60) arises from $\boldsymbol{a}^{\mathrm{T}} M \boldsymbol{a} + \boldsymbol{l} \cdot \boldsymbol{a} = 0$ on neglecting the terms of second order in $\boldsymbol{a}$.

Let $\boldsymbol{a}$ have direction $(1, 1, -1)^{\mathrm{T}}$. The last equation of (60) yields $l_3 = l_1 + l_2$. On substituting for l_3, the first two equations of (60) yield

$$\left.\begin{aligned} 2p_{43}(ep_{41} + fp_{42}) + l_1(p_{41} + p_{43}) + l_2(p_{42} + p_{43}) &= 0, \\ 2p_{53}(ep_{51} + fp_{52}) + l_1(p_{51} + p_{53}) + l_2(p_{52} + p_{53}) &= 0. \end{aligned}\right\} \tag{61}$$

The equation $\boldsymbol{l}^{\mathrm{T}}N\boldsymbol{l} = 0$ reduces to

$$(l_1 + l_2)(el_1 + fl_2) = 0. \tag{62}$$

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It follows from (61) and (62) by direct calculation that for almost all choices of $\boldsymbol{p}_4$, $\boldsymbol{p}_5$ there exist three distinct solutions (d, e, f) in $\mathbb{P}^3$, obtained as functions of $\boldsymbol{p}_4$ and $\boldsymbol{p}_5$. (Note $d = 0$.) The remaining two cases $e = 0$ and $f = 0$ also each yield three distinct solutions for (d, e, f) for almost all choices of $\boldsymbol{p}_4$, $\boldsymbol{p}_5$. Thus, in general, a total of nine distinct intersections of the two cubic plane curves is obtained. □

In theorem 8.4 the possibility is not ruled out that the intersections of the two cubic plane curves yield quadrics without real generators. Quadrics obtained in this way give rise to complex camera displacements compatible with the image correspondences, but which are not physically acceptable.

9. Image velocities

There is an alternative formulation of the reconstruction problem based on image velocities rather than image correspondences (Horn 1986, 1987; Longuet-Higgins & Prazdny 1980; Maybank 1985, 1987). In this formulation the relative positions of points fixed in space are deduced from the velocities of the images of these points due to motion relative to the camera. Reconstruction based on image velocities is thus a limiting case of reconstruction based on image correspondences as the displacement of the camera becomes small.

There are many similarities between the mathematical structure of the two approaches to reconstruction. For example, the ambiguous surfaces associated with image velocities are rectangular hyperboloids, and there are, in general, exactly ten rigid velocities compatible with five given image velocities (Maybank 1989). The question of the extent of the similarities between the two approaches now arises. As a partial answer, it is shown that theorem 7.1 does not carry over to the case of image velocities. In spite of this negative result, the ambiguous surfaces arising from image velocities are subject to two cubic polynomial constraints analogous to those obtained in §7.3 for the case of image displacements.

The mathematical framework for reconstruction based on image velocities is as follows. Let the camera obtain an image of a rigid surface moving with velocity $\{\boldsymbol{v}, \boldsymbol{w}\}$, where $\boldsymbol{v}$ is the translational velocity, and $\boldsymbol{w}$ is the angular velocity taken about an axis passing through the optical centre of the camera (Sokolnikoff & Redheffer 1966). Maybank (1987) shows that if the image is formed by polar projection onto a spherical projection surface then the velocities $\dot{\boldsymbol{q}}$ of image points $\boldsymbol{q}$ are given by

$$\dot{\boldsymbol{q}} = [\boldsymbol{v} - (\boldsymbol{v}\cdot\boldsymbol{q})\,\boldsymbol{q}]\,K + \boldsymbol{w}\times\boldsymbol{q}, \tag{63}$$

where $1/K$ is the distance in the direction $\boldsymbol{q}$ from the optical centre of the camera to the surface.

9.1. *Image velocities and ambiguous surfaces*

Ambiguous surfaces arise in the case of image velocities just as in the case of image displacements. Let ψ be a rigid surface moving with a velocity $\{\boldsymbol{w}_1, \boldsymbol{v}_1\}$ relative to the camera such that the resulting image velocities are compatible with a second rigid surface ϕ moving with a velocity $\{\boldsymbol{w}_2, \boldsymbol{v}_2\}$, such that $\boldsymbol{v}_2$ is not parallel to $\boldsymbol{v}_1$. Let cartesian coordinates be chosen with origin $\boldsymbol{o}$ at the optical centre of the camera. Then ψ and ϕ are given respectively by the equations

$$(\boldsymbol{w}\cdot\boldsymbol{x})(\boldsymbol{v}_2\cdot\boldsymbol{x}) - (\boldsymbol{w}\cdot\boldsymbol{v}_2)(\boldsymbol{x}\cdot\boldsymbol{x}) + (\boldsymbol{v}_1\times\boldsymbol{v}_2)\cdot\boldsymbol{x} = 0, \tag{64}$$

$$(\boldsymbol{w}\cdot\boldsymbol{x})(\boldsymbol{v}_1\cdot\boldsymbol{x}) - (\boldsymbol{w}\cdot\boldsymbol{v}_1)(\boldsymbol{x}\cdot\boldsymbol{x}) + (\boldsymbol{v}_1\times\boldsymbol{v}_2)\cdot\boldsymbol{x} = 0, \tag{65}$$

where $\boldsymbol{w} \equiv \boldsymbol{w}_2 - \boldsymbol{w}_1$. As in the case of image displacements, ϕ and ψ are rectangular hyperboloids. The intersection $\phi \cap \psi$ is the union of a horopter curve and a common generator of ψ and ϕ. In addition, $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ can be scaled such that both are on the horopter curve. The vector $\boldsymbol{v}_2$ lies on a generator of ψ passing through $\boldsymbol{o}$, and $\boldsymbol{v}_1$ lies on a generator of ϕ passing through the $\boldsymbol{o}$. The quadrics ψ and ϕ are said to be complementary ambiguous surfaces. The following theorem shows that in the case of image velocities there is no theorem analogous to theorem 7.1.

Theorem 9.1. *Let ψ, ϕ be complementary ambiguous surfaces compatible with the same image velocities, let τ_ψ be the unique non-trivial rigid skew involution of ψ that fixes the principal points of ψ, and let τ_c be the unique non-trivial rigid involution of the horopter curve c contained in $\psi \cap \phi$. Let ψ and the image velocities be fixed, and let ϕ and hence c vary. Then, in general, $\tau_\psi \neq \tau_c$ for all choices of ϕ.*

Proof. Let cartesian coordinates be chosen with origin $\boldsymbol{o}$ at the optical centre of the camera, and let $\boldsymbol{m}$, $\boldsymbol{n}$ be the principal points of ψ, chosen such that $\boldsymbol{n}$ is also a principal point of ϕ. Let ψ, ϕ have equations (64), (65) respectively. Then $\langle \boldsymbol{o}, \boldsymbol{n} \rangle$ is the line through $\boldsymbol{o}$ with direction $\boldsymbol{w}$, and $\langle \boldsymbol{o}, \boldsymbol{m} \rangle$ is the line through $\boldsymbol{o}$ with direction $\boldsymbol{v}_2$. The line $\langle \boldsymbol{o}, \boldsymbol{m} \rangle$ is a generator of ψ, whereas $\langle \boldsymbol{o}, \boldsymbol{n} \rangle$ is, in general not a generator of ψ. The variation of ϕ referred to in the statement of the theorem is obtained by changing the length of $\boldsymbol{v}_2$. Suppose, if possible, that for some choice of ϕ, $\tau_\psi = \tau_c$. A contradiction is obtained as follows.

The line $\langle \boldsymbol{o}, \boldsymbol{m} \rangle$ meets c at just one point, namely the centre $\boldsymbol{c}$ of c. Let Π be the common tangent plane to ψ and ϕ at $\boldsymbol{o}$. Then $\Pi \cap \phi$ consists of two lines meeting at $\boldsymbol{o}$. Now $\Pi \cap \phi$ contains both $\boldsymbol{o}$ and $\boldsymbol{c}$. If $\boldsymbol{c} \neq \boldsymbol{o}$ then the line $\langle \boldsymbol{o}, \boldsymbol{c} \rangle = \langle \boldsymbol{o}, \boldsymbol{m} \rangle$ is contained in ϕ. Hence $\boldsymbol{v}_2$ defines a principal point of ϕ. It follows that

$$(\boldsymbol{w} \cdot \boldsymbol{v}_2)(\boldsymbol{v}_1 \cdot \boldsymbol{v}_2) - (\boldsymbol{w} \cdot \boldsymbol{v}_1)(\boldsymbol{v}_2 \cdot \boldsymbol{v}_2) = 0. \tag{66}$$

In general (66) does not hold. Thus $\boldsymbol{o} = \boldsymbol{c}$. It follows that $\langle \boldsymbol{o}, \boldsymbol{n} \rangle$ is a generator of ψ, thus

$$\boldsymbol{w} \cdot (\boldsymbol{v}_1 \times \boldsymbol{v}_2) = 0. \tag{67}$$

In general (67) does not hold. Thus the hypothesis $\tau_c = \tau_\psi$ does not hold. □

9.2. *Cubic constraints on ambiguous surfaces*

Let equation (64) for the ambiguous surface ψ be written in the form

$$\boldsymbol{x}^{\mathrm{T}} M' \boldsymbol{x} + \boldsymbol{l}' \cdot \boldsymbol{x} = 0,$$

where M' is a 3×3 symmetric matrix and $\boldsymbol{l}'$ is a vector. Define the matrix N' by

$$N' = M' - \tfrac{1}{2} \operatorname{tr}(M') I.$$

The cubic constraints analogous to (56) and (57), which apply to ψ are

$$\boldsymbol{l}'^{\mathrm{T}} N' \boldsymbol{l}' = 0, \tag{68}$$

$$\det(N') = 0. \tag{69}$$

Equations (68) and (69) are readily obtained from (64). Equation (69) is given by Horn (1987).

On comparing (68) and (56), it can be seen that (68) is the limit of (56) as the translation of the camera tends to zero, and simultaneously the rotation of the camera tends to the identity. Similarly (69) is the limit of (57).

10. Conclusion

The reconstruction of the relative positions of points in space from the correspondences between two different images is subject to ambiguity if the points lie on certain surfaces of degree two. The ambiguous case of reconstruction has been investigated using the projective geometric methods developed by the photogrammetrists during the last century and the first half of this century. These methods have not so far been exploited in computer vision because they are expressed in a mathematical framework which has fallen out of fashion, and because the literature in this area is not widely known. This paper is in part an attempt to advertise the projective geometric approach. Projective geometry is a theory of considerable elegance, and the results obtained from it give additional insight into ambiguity.

Two possible lines for future work are suggested. The first is to clarify the connections between ambiguity and the general problem of stability of reconstruction. The two cubic constraints of §7.3 may provide a good starting point. It is conjectured that instability of reconstruction arises because the two cubic constraints are near coincident over a wide range of their domain of definition. The second and related line of research is to investigate the behaviour of algorithms for reconstruction from image correspondences in the presence of noise as the distances between corresponding points becomes small.

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